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On the continuous and discontinuous maximal operators

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ABSTRACT

In the first part of this paper we study the regularity properties of a wide class of maximal operators. These results are used to show that the spherical maximal operator is continuous $W^{1,p}(\mathbb{R}^n) \mapsto W^{1,p}(\mathbb{R}^n)$, when $p > \frac{n}{n-1}$. Other given applications include fractional maximal operators and maximal singular integrals. On the other hand, we show that the restricted Hardy–Littlewood maximal operator M_λ, where the supremum is taken over the *cubes* with radii greater than $\lambda > 0$, is bounded from $L^p(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ but *discontinuous*.
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1. Introduction

The research of the regularity properties of the maximal operators was launched by J. Kinnunen [8] when he observed that the Hardy–Littlewood maximal operator is bounded in $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. After that, several works have been devoted to this topic. We mention here [1,3,7,9–13] and [17].

For nonlinear operators, like maximal operators, it is important to notice that in general the continuity is not implied by the boundedness. It is accordingly natural to ask if the maximal operator is also continuous in Sobolev spaces. This question was posed in [7], where it was attributed to T. Iwaniec, and the positive answer was given by the author in [12]. The proof included new techniques based on a careful analysis of the set of radii for which the maximal average is attained.

In the present paper our first goal is to generalize the results and the technology developed in [12] for a class of maximal operators acting on possibly different function spaces. The method introduced in [12] has turned out to be applicable for various maximal operators (for example, see [5] and [4]). This gives a natural motivation for the study of a more general theory. The second main focus of this work is to enlight the continuity problems from the opposite point of view: Do there exist natural maximal operators which are bounded but discontinuous from a function space to a Sobolev space?

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1.1. Positive results for a class of maximal operators

We are going to prove results for a maximal operator T^* determined by a family $\{T_r\}_{r\in I}$ of linear operators acting on $L^p(\mathbb{R}^n)$, and defined by

$$T^*f(x) = \sup_{r\in I} \big|T_r f(x)\big|. \tag{1}$$

The usual maximal operators are of this type, as well as some operators which are not always called maximal operators (like the maximal Hilbert transform). For us this level of generality is a natural starting point. We assume that T^* is bounded in L^q (for some $1 < q < \infty$), and then we ask what are the minimal assumptions guaranteeing the existence and desired properties for the weak derivative of T^*f.

Due to the generality of the treatment, we face many technical problems, which does not occur, for example, in the case of the standard Hardy–Littlewood maximal operator. The first problem concerns already the validity of the definition (1): we have to make sure that for all f in the domain of T^* it holds for almost every x that the values $T_r f(x)$ on the right-hand side of (1) are defined for every r. Indeed, each $T_r f$ has to be a real function, not an equivalence class of functions.

The second problem is closely related to the first problem. It arises from our desire to write

$$D_i(T^*f)(x) = T_r(D_i f)(x)\,, \tag{2}$$

for all $r \in I$ such that $T^*f(x) = T_r f(x)$. This would be a counterpart of the formula for the derivative of the Hardy–Littlewood maximal function, which is the main tool in [12] and an interesting result in itself, as well. The first problem arising is that for the validity of (2) it is essential that $T_r f(x)$ is (a.e.) continuous with respect to r (see [12]). The second problem derives from our desire to achieve a theory, which does not require the existence of $D_i f$ (this is the situation e.g. with the fractional maximal operator, Theorem 3.3). Therefore, (2) is not an appropriate candidate for the formula for $D_i(T^*f)$. The next candidate is naturally

$$D_i(T^*f)(x) = D_i(T_r f)(x) \tag{3}$$

for all $r \in I$ such that $T^*f(x) = T_r f(x)$. This is actually very close to what we are going to prove. Instead of assuming that $D_i f$ exists, we require that $D_i(T_r f)$ exists for all $r \in I$. Observe that for usual maximal operators the corresponding operators T_r are smoothing (averaging operators), thus assuming the existence of $D_i(T_r f)$ is much less restrictive than assuming the existence of $D_i f$. However, again some technical problems arise, since it would be too restrictive to require $T_r f$ to be differentiable in the classical sense. For example, this is the situation with two of our applications (Theorems 3.3 and 3.1). This means serious problems for (3): In the worst case it may happen that for all x it holds that for $r = r(x)$ such that $T^*f(x) = T_r f(x)$, $D_i(T_r f)(x)$ is not defined.

In the problems explained above the key point is that *a priori* the exceptional sets of measure zero for every single $r \in I$ may cause serious problems for the validity of the definition of T^* and formula (3). If we assumed that I is countable, we could avoid the problems concerning the definition of T^*. However, this assumption would cause other technical and notational problems.

To achieve the desired results, we will assume that family $\{T_r\}$ is *admissible*, i.e. it satisfies some natural properties which are given in (A1)–(A7) in the beginning of Section 2. Under these assumptions we prove that boundedness of T^* from $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ implies the boundedness and continuity of T^* from $\mathcal{F}_1$ to $W^{1,q}(\mathbb{R}^n)$, where $\mathcal{F}_1 \subset L^p(\mathbb{R}^n)$ is a suitable normed function space given in (A2) (typically a Sobolev space or a Lebesgue space).

The most important requirement from $(A1), \ldots, (A7)$ seems to be that operator R^*, defined by

$$R^*f(x) = \sup_{r\in I_0, 1\le i\le n} |D_i T_r f(x)|,$$

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