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Solvability of one non-Newtonian fluid dynamics model with memory

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A B S T R A C T

In the present paper we establish the existence of weak solutions to the initial–boundary value problem for one viscoelastic model of Oldroyd's type fluid with memory along trajectories of the velocity field. Previously such problem has been studied for corresponding regularized models. The reason of the regularization was the lack of results on the solvability of the Cauchy problem with not sufficiently smooth velocity field. However, recent results about regular Lagrangian flows (generalization of classical solutions of a Cauchy problem) allow to establish the existence theorem for the original problem. We use topological approximation method which involves the approximation of the original problem by regularized operator equations with consequent application of topological degree theory for its solvability. This allows to establish the existence of weak solutions of considered problem on the base of a priori estimates and passing to the limit.

1. Introduction

We consider the motion of a fluid that occupies a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with locally Lipschitz boundary $\partial\Omega$ on a time interval $[0,T]$, $T > 0$. The motion equation in the Cauchy form (see [11], Ch. II, Sec. 4–6) is

$$\rho(\partial v(t,x)/\partial t + \sum_{i=1}^{n} v_i(t,x)\,\partial v(t,x)/\partial x_i) =$$
$$-\nabla p(t,x) + \operatorname{Div}\sigma(t,x) + \rho f(t,x),\ (t,x) \in Q = [0,T] \times \Omega, \quad (1.1)$$

where $v = (v_1(t,x), \dots, v_n(t,x))$ is the velocity at a point $x \in \Omega$ at time t; ρ is the fluid density; $p = p(t,x)$ is the pressure; $\sigma = \sigma(t,x) = \{\sigma_{ij}(t,x)\}_{i,j=1}^{n}$ is the deviator of the stress tensor; $f = f(t,x)$ is the density

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of external forces; $\mathrm{Div}\,\sigma$ is a vector function whose coordinates are divergences of the rows of the matrix σ. Below ρ supposed to be equal to 1 for simplicity.

Eq. (1.1) is completed by a constitutive law (rheological relation) defining the type of a fluid (see [16]).

The constitutive law $\sigma = 2\nu\mathcal{E}(v)$ defines the Newtonian fluid. Here $\mathcal{E}(v) = \{\mathcal{E}_{ij}\}_{i,j=1}^n$, $\mathcal{E}_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ is the strain rate tensor. The well known Navier–Stokes system corresponds to this law. However, there is a wide range of viscous incompressible non-Newtonian fluids (see e.g. [1,4,5,12,15,22,23,25,26]), among them are Maxwell, Kelvin–Voigt, Oldroyd and other models. One of the important non-Newtonian fluid is determined by the rheological relation

$$(1 + \lambda\, d/dt)\sigma = 2\nu(1 + \varkappa\nu^{-1} d/dt)\mathcal{E}(v) \tag{1.2}$$

where $d/dt = \partial/\partial t + \sum_{i=1}^n v_i\partial/\partial x_i$ is the total derivative and $\lambda, \varkappa, \nu$ are positive constants.

Fluids of (1.2) type have been introduced and extensively studied by Jeffreys and Oldroyd (see e.g. [12,15]). Integrating (1.2) along the velocity field v, solving it with respect to σ and substituting the result in Eq. (1.1) we get the following Jeffreys–Oldroyd initial–boundary value problem

$$\partial v(t,x)/\partial t + \sum_{i=1}^n v_i(t,x)\partial v(t,x)/\partial x_i - \mu_0 \Delta v(t,x) - \tag{1.3}$$

$$\mu_1 \mathrm{Div} \int_0^t \exp((s-t)/\lambda)\,\mathcal{E}(v)(s, z(s;t,x))ds + \nabla p(t,x) = f(t,x),\ (t,x) \in Q;$$

$$\mathrm{div}\, v(t,x) = 0, \quad (t,x) \in Q; \tag{1.4}$$

$$z(\tau;t,x) = x + \int_t^\tau v(s, z(s;t,x))\, ds, \quad 0 \leqslant t, \tau \leqslant T, \quad x \in \overline{\Omega}; \tag{1.5}$$

$$v(0,x) = v^0(x),\ x \in \Omega; \quad v(t,x) = 0, \quad (t,x) \in \Gamma = [0,T] \times \partial\Omega. \tag{1.6}$$

Here $\mu_0 = 2\varkappa$, $\mu_1 = 2(\nu - \varkappa)$ are constitutive constants. Details can be found in [26], Sec. 7.1.

Survey of results on mathematical problems for Jeffreys–Oldroyd models is given in [22]. The integral term in (1.3) is related to the memory of the fluid along trajectories of the velocity field v. Note, that the system (1.3)–(1.6) contains not only unknown velocity v and pressure p, but also the trajectory $z(\tau;t,x)$ being defined by the Cauchy problem (in the integral form) (1.5).

Different models with memory have been studied in many papers (see. e.g. [1,4,5,10,13,14,16,22–26] et al.). Weak solvability of problem (1.3)–(1.6) where v in (1.5) is replaced by its smooth regularization $\widetilde{v}$ have been considered in [23]. The reason of the regularization is the impossibility to define a unique classical solution to the Cauchy problem (1.5) for v from the class of weak solutions of problem (1.3)–(1.6). Relatively recent results on a solvability of Cauchy problem (1.5) for "low" regular v in the class of regular Lagrangian flows (generalization of the concept of a classical solution) allow to establish existence, uniqueness and stability of solutions to problem (1.5) in mentioned class (see e.g. [2,6–8]). This allows to get the existence of weak solutions to the problem (1.3)–(1.6) without a regularization of v in Eq. (1.5). For this purpose results of [2,6–8,23] have been substantially used.

The paper is organized as follows. Section 2 provides basic notations, auxiliary statements and the statement of the main result. In Section 3 we introduce a family of two-parameterized regularized problems for (1.3)–(1.6). In Section 4 we prove the solvability of regularized problems. For this we reformulate these problems in the form of operator equations (Section 4.1) and use the topological degree theory for the solvability of these equations (Section 4.3). The solvability of regularized problems (1.3)–(1.6) is established in Section 4.4. In Section 5 we obtain estimates of solutions of regularized problems. Section 6 is devoted to the proof of the main Theorem 2.3. For this we construct a family of approximative regularized problems (Section 6.1) and on the base of results of Sections 4 and 5 we establish the solvability of approximative