Nonlinear Analysis 172 (2018) 99–114

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

A heteroclinic orbit connecting traveling waves pertaining to different nonlinearities in a channel with decreasing cross section

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ARTICLE INFO

Article history:
Received 23 November 2017
Accepted 5 March 2018
Communicated by Enzo Mitidieri

Keywords:
Reaction diffusion equation
Traveling wave
Heteroclinic orbit

ABSTRACT

In this paper we consider a semilinear parabolic equation in an infinite cylinder with shrinking cross section. The portion where the domain is not a straight half-cylinder is assumed to be compact. The spatially varying nonlinearity is such that it connects two (spatially independent) bistable nonlinearities in a compact set in space. We prove that, given such a setting, a traveling wave obeying the equation with the one bistable nonlinearity and starting at the respective side of the decreasing cylinder, will converge to a traveling wave solution prescribed by the nonlinearity on the other side.

1. Introduction

Since the pioneering works of Kolmogorov et al. [10] the study of traveling wave solutions for semilinear parabolic equations of several types (the most prominent are bistable, ignition and KPP-type nonlinearities) has been an active field. In the case of bistable nonlinearities, that we will concern ourselves with in this article, we want to refer to the celebrated paper by Fife and McLeod [8] for existence and uniqueness as well as stability in one spatial dimension. We also want to mention the detailed study of traveling waves in cylinders by Berestycki and Nirenberg [5]. Later Berestycki, Hamel et al. have broadened the field by studying generalizations of traveling waves in domains or with coefficients/nonlinearities that do not allow for traveling wave solutions. In the case of periodic media, this has led to the notion of pulsating fronts [2] and recently they have generalized it to the notion of transition fronts that do not require any special properties of the domain (apart from sufficiently smooth boundary and infinite geodesic diameter) or of the coefficients [3]. In this respect we found [4] very inspiring where Matano, Berestycki and Hamel use super- and subsolutions as in [11] to construct in an outer domain an entire solution that starts as a traveling wave solution for $t \to -\infty$, passes the obstacle and converges – given the compact obstacle is sufficiently regular – to the same traveling wave solution as $t \to +\infty$. Another recent and very interesting work on the construction of generalized transition fronts is [12] where the author studies generalized transition fronts

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https://doi.org/10.1016/j.na.2018.03.004

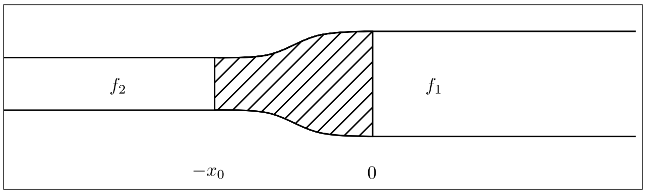

Fig. 1. Infinite cylinder with transition zone and shrinking diameter.

in cylinders subject to a space dependent nonlinearity that is bounded from above and below by spatially independent ignition-type nonlinearities. We are trying to investigate a similar problem as investigated in [12], but in our case the nonlinearities are of bistable type and do only vary in a compact transition zone. Furthermore this work has been inspired by the analysis of front blocking and propagation [1] where the authors prove propagation and calculate the limit speed of a wave that passes through a shrinking cylinder similar to ours but subject to a constant in space nonlinearity. In contrast to [12,1] we are interested in the existence of a heteroclinic connection between two traveling fronts, which is stronger than the relaxed notion of a transition front (as given in [3]).

In this paper we will occupy ourselves with the construction of a transition front in a shrinking pipe, i.e. $D \subset \mathbb{R}^n$ shall be such that

$$D^+ := D \cap \{x_1 \geq 0\} = [0, \infty) \times \Omega_1, \text{ where } \Omega_1 \subset \mathbb{R}^{n-1} \text{ is a smooth domain,}$$
$$D^- := D \cap \{x_1 \leq -x_0\} = (-\infty, -x_0] \times \Omega_2, \text{ where } \Omega_2 \subset \mathbb{R}^{n-1} \text{ is a smooth domain,}$$
$$\partial D \text{ is smooth and the outer normal satisfies } \nu \cdot e_1 \leq 0,$$

and $x_0 > 0$ is a parameter of the transition region (see Fig. 1).

Thanks to the compactness of the transition region, we can construct a heteroclinic orbit between two traveling wave solutions. To be more precise the nonlinearity $f(x, u)$ shall be such that

$$\begin{cases} f_2(u) \leq f(x,u) \leq f_1(u) \text{ for } x \in D, u \in [0,1], \\ f(x,u) = f_1(u) \text{ for } x_1 \geq 0, u \in [0,1] \text{ and} \\ f(x,u) = f_2(u) \text{ for } x_1 \leq -x_0, u \in [0,1], \end{cases} \tag{1.1}$$

where f_1, f_2 are two a-priori given nonlinearities of bistable type.

The main result of this article shall be

Theorem 1.1. *Let f satisfy (1.1) (and (2.1)–(2.4)) then there is a unique entire solution $u(t, x)$ of*

$$\begin{cases} \partial_t u - \Delta u = f(x,u) & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D, \end{cases} \tag{1.2}$$

such that $0 < u(t, x) < 1$ and $\partial_t u(t, x) > 0$ for all $(t, x) \in \mathbb{R} \times \bar{D}$ and

$$u(t,x) - \phi_1(x_1 + c_1 t) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \bar{D},$$

as well as

$$u(t,x) - \phi_2(x_1 + c_2 t + \beta) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \bar{D}$$

for some $\beta \in \mathbb{R}$.

Here (ϕ_1, c_1) and (ϕ_2, c_2) are the one-dimensional traveling wave profiles and corresponding speeds solving for $i \in \{1, 2\}$