Nonlinear Analysis 172 (2018) 216–232

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Global $W^{2,\delta}$ estimates for singular fully nonlinear elliptic equations with L^n right hand side terms[☆]

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ARTICLE INFO

Article history:
Received 13 June 2017
Accepted 12 March 2018
Communicated by Enzo Mitidieri

MSC:
35B45
35D40
35J60
35J75

Keywords:
Fully nonlinear elliptic equations
Global $W^{2,\delta}$ estimates
Singular elliptic equations

ABSTRACT

We establish in this paper *a priori* global $W^{2,\delta}$ estimates for singular fully nonlinear elliptic equations with L^n right hand side terms. The method is to slide paraboloids and barrier functions vertically to touch the solution of the equation, and then to estimate the measure of the contact set in terms of the measure of the vertex point set. To derive global estimates from L^n data, the Hardy–Littlewood maximal functions, appropriate localizations and a new type of covering argument are adopted. These methods also provide us a more direct proof of the $W^{2,\delta}$ estimates for (nonsingular) fully nonlinear elliptic equations established by L. A. Caffarelli and X. Cabré.

1. Introduction

In the present paper, we derive *a priori* global $W^{2,\delta}$ estimates for solutions of the singular elliptic equations including those of the following types:

(a) the equation

$$\operatorname{tr}(A(x)D^2u) + b(x)\cdot Du = f(x)|Du|^{\gamma}, \tag{1.1}$$

where $0 \le \gamma < 1$, A is uniformly elliptic, b is bounded and $f \in L^n$;

(b) the singular fully nonlinear elliptic equation

$$|Du|^{-\gamma}F(D^2u, Du, u, x) = f(x), \tag{1.2}$$

where $0 \le \gamma < 1$, $F(0,0,\cdot,\cdot) \equiv 0$, F is uniformly elliptic (see [9]), and $f \in L^n$;

☆ This research is partially supported by NSFC 11671316.
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https://doi.org/10.1016/j.na.2018.03.010
0362-546X/

(c) the famous p-Laplace equation

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2}Du\right) = f(x), \tag{1.3}$$

where $1 < p \leq 2$ and $f \in L^n$.

For brevity, we consider solutions of singular fully nonlinear elliptic inequalities of certain type which include solutions of all the above equations. Namely, our main result will be stated in a more generalized form as follows.

Theorem 1.1. *Let $0 < \lambda \leq \Lambda < +\infty$ and $0 \leq \gamma < 1$. Suppose $u \in C^0(\overline{B}_1)$ is a viscosity solution of the singular fully nonlinear elliptic inequalities*

$$|Du|^{-\gamma}\mathcal{P}^-_{\lambda,\Lambda}(D^2u) - |Du|^{1-\gamma} \leq f \leq |Du|^{-\gamma}\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + |Du|^{1-\gamma} \quad \text{in } B_1, \tag{1.4}$$

where B_1 is the unit open ball of $\mathbb{R}^n$, $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ are the Pucci extremal operators, and $f \in C^0 \cap L^n(B_1)$. Then $u \in W^{2,\delta}(B_1)$ and

$$\|u\|_{W^{2,\delta}(B_1)} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|^{\frac{1}{1-\gamma}}_{L^n(B_1)}\right) \tag{1.5}$$

for any $\delta \in (0, \sigma)$, where $\sigma = \sigma(n, \lambda, \Lambda, \gamma) > 0$ and $C = C(n, \lambda, \Lambda, \gamma, \delta) > 0$.

This theorem improves essentially our previous results in [16], where the right hand side term f of the equation is required to be L^∞, and the estimate corresponding to (1.5) is just

$$\|u\|_{W^{2,\delta}(B_1)} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|^{\frac{1}{1-\gamma}}_{L^\infty(B_1)}\right).$$

The main contribution here is in developing a systematic way to deal with the L^n data, and in using delicate localization and covering arguments to derive global estimates from it in a straightforward way. Roughly speaking, first, by sliding paraboloids and some appropriate localizing barrier functions from below and above to touch the solution, and then estimating the low bound of the measure of the set of contact points by the measure of the set of vertex points, we establish a new density estimate which is corresponding to the classical Alexandroff–Bakelman–Pucci (ABP for short) estimate; then, applying a new kind of covering technique with careful localization, we obtain the desired global $W^{2,\delta}$ estimates. These methods also provide us a more direct proof of the interior $W^{2,\delta}$ estimates for (nonsingular) fully nonlinear elliptic equations established by L.A. Caffarelli and X. Cabré [9] (which now is recovered by Theorem 1.1 as special cases, rather than by our previous results in [16]), although the underlying key ideas are the same.

The sliding paraboloid argument we mentioned above has originated in the work of X. Cabré [7] and continued in the work of O. Savin [18], see also [15,10] and [11]. We now give some other historical remarks concerning the $W^{2,\delta}$ estimates and the singular elliptic equations.

In 1986, F.-H. Lin [17] first established the $W^{2,\delta}$ estimates $\left\|D^2u\right\|_{L^\delta(B_1)} \leq C\,\|f\|_{L^n(B_1)}$ for solutions of the linear uniformly elliptic equations $a_{ij}(x)u_{ij} = f(x)$ with $u = 0$ on ∂B_1 and $f \in L^n(B_1)$. His method employs the Fabes–Stroock type reverse Hölder inequality, estimates for Green's function and the ABP estimate. Later, L.A. Caffarelli and X. Cabré [9] (see also [8]) applied ABP estimate, Calderón–Zygmund cube decomposition technique, barrier function method and touching by tangent paraboloid method to obtain interior $W^{2,\delta}$ estimates for viscosity solutions of the fully nonlinear elliptic inequalities

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u) \leq f \leq \mathcal{P}^+_{\lambda,\Lambda}(D^2u),$$