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Two-weight characterization for commutators of bi-parameter fractional integrals

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ABSTRACT

Let $\mathcal{I}_{\vec{\alpha}}$ be the bi-parameter fractional integral operator on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\mathcal{I}_{\vec{\alpha}}(f)(x) = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \frac{f(y_1, y_2)}{|x_1 - y_1|^{n_1 - \alpha_1} |x_2 - y_2|^{n_2 - \alpha_2}} dy, \quad 0 < \alpha_i < n_i, \ i = 1, 2.$$

In this paper, we give a characterization of two-weight norm inequality for the commutator of $\mathcal{I}_{\vec{\alpha}}$. We show that for $\mu, \lambda \in A_{p,q}(\mathbb{R}^{\vec{n}})$, $\|[b, \mathcal{I}_{\vec{\alpha}}]\|_{L^p(\mu^p) \to L^q(\lambda^q)} \simeq \|b\|_{bmo(\nu)}$, where $\nu = \mu\lambda^{-1}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha_1}{n_1} = \frac{\alpha_1}{n_2}$. It extends the recent one-parameter theory to the bi-parameter setting. We use the modern dyadic methods, in which the main idea is to represent continuous operators in terms of dyadic operators. Moreover, by introducing some new full and mixed bi-parameter dyadic paraproducts, we write the commutator as a finite linear combination of them.

1. Introduction

In the multi-parameter setting, characterizations of BMO spaces through boundedness of commutators have been the subject of many recent research papers in harmonic analysis. It originated in the work of Ferguson and Sadosky [5], Ferguson and Lacey [4]. The authors respectively obtained the upper and lower bound for bi-parameter iterated commutators of Hilbert transform. Then the multi-parameter result was established by Lacey and Terwilleger [14]. Along this way, soon after, the authors in [12,13] and [3] gave the corresponding conclusions for Riesz transforms and a large class of Calderón–Zygmund operators. Recently, the product BMO space was restudied and characterized in [2,16], where they presented a new proof of boundedness of iterated commutators, using Hytönen's representation theorem [10]. Significantly, new methods using Journé operators in [16] were developed to obtain lower norm estimates in the multi-parameter real variable setting.

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It is well known that the fractional integral operator I_α was introduced by M. Riesz

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

After that, its many applications in analysis were found. For example, the research to the fractional integral operator is closely related to the theory of Sobolev space and the Laplacian operator of fractional degree. Later on, the commutator of the fractional integral operator was first studied by Chanillo [1]. However, in the recent paper [11] by Lacey, he presented a new dyadic proof. Beyond that, Lacey's methods are valid in arbitrarily many parameters. In contrast to typical methods addressing multi-parameter problems, it enables the argument to be iterated. For more publications about the fractional integrals, the readers can refer to [9,15] and [17].

In this paper, we present a new characterization of the weighted little product BMO space $bmo(\nu)$ in terms of commutators of the bi-parameter fractional integral operator $\mathcal{I}_{\vec{\alpha}}$, which is defined by

$$\mathcal{I}_{\vec{\alpha}}(f)(x) = \int_{\mathbb{R}^{\vec{n}}} \frac{f(y_1, y_2)}{|x_1-y_1|^{n_1-\alpha_1}|x_2-y_2|^{n_2-\alpha_2}} dy, \quad 0 < \alpha_i < n_i.$$

Motivated by the works of [11] and [9], we also want to use the modern dyadic methods. To do this, we first introduce the dyadic model operator of $\mathcal{I}_{\vec{\alpha}}^{\mathcal{D}}$ as follows.

$$\mathcal{I}_{\vec{\alpha}}^{\mathcal{D}}(f)(x) := \sum_{Q_1 \times Q_2 \in \mathcal{D}} |Q_1|^{\frac{\alpha_1}{n_1}} |Q_2|^{\frac{\alpha_2}{n_2}} \langle f \rangle_{Q_1 \times Q_2} \cdot \mathbb{1}_{Q_1 \times Q_2}(x), \ x \in \mathbb{R}^{\vec{n}},$$

where $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ and $\mathcal{D}_i$ is a dyadic grid on $\mathbb{R}^{n_i}$, $i = 1, 2$. We obtain the fact that $\mathcal{I}_{\vec{\alpha}}$ can be recovered from $\mathcal{I}_{\vec{\alpha}}^{\mathcal{D}}$ by averaging over dyadic grids. In our proof, we will reduce the original problem to the upper bound for commutators with dyadic operators. There are many recent developments about this ideology (cf. e.g. [6–9]).

The main result of this paper is the following.

Theorem 1.1. *Let $0 < \alpha_i < n_i$, $1 < p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha_i}{n_i}$, $i = 1, 2$. Suppose that weights $\mu, \lambda \in A_{p,q}(\mathbb{R}^{\vec{n}})$ and $\nu = \mu\lambda^{-1}$. Then it holds that*

$$\|[b, \mathcal{I}_{\vec{\alpha}}]\|_{L^p(\mu^p) \to L^q(\lambda^q)} \simeq \|b\|_{bmo(\nu)}.$$

The article is organized as follows. In Section 2, we present some necessary definitions and notation, such as Haar functions, $A_{p,q}$ weights, and weighted product BMO spaces. In Section 3, we introduce varieties of bi-parameter dyadic paraproducts. Then two-weight inequalities for them are established. In Section 4, we show that $\mathcal{I}_{\vec{\alpha}}$ can be recovered from $\mathcal{I}_{\vec{\alpha}}^{\mathcal{D}}$ by averaging over dyadic grids. Then the proof of the upper bound is reduced to the dyadic version $[b, \mathcal{I}_{\vec{\alpha}}^{\mathcal{D}}]$. To finish this, we write the commutator as a finite linear combination of bi-parameter paraproducts. Finally, in Section 5, we will directly show the lower bound for the commutator $[b, \mathcal{I}_{\vec{\alpha}}]$.

2. Preliminaries

In order to show our results, we here present some definitions and lemmas.

2.1. Random dyadic grids

Let $\mathcal{D}_0$ be the standard dyadic grid on $\mathbb{R}^n$. That is,

$$\mathcal{D}_0 := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_0^k, \quad \mathcal{D}_0^k := \{2^k([0,1)^n + m);\ m \in \mathbb{Z}^n\}.$$