Nonlinear Analysis 171 (2018) 73–84

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On nonexistence of global solutions for a semilinear heat equation on graphs

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ARTICLE INFO

Article history:
Received 17 August 2017
Accepted 25 January 2018
Communicated by Enzo Mitidieri

MSC:
35A01
35K91
35R02
58J35

Keywords:
Semilinear heat equation
Heat kernel estimate
Graphs

ABSTRACT

Let $G = (V, E)$ be a simple, finite, connected, weighted graph satisfying curvature condition $CDE'(n, 0)$ and polynomial volume growth $\mathcal{V}(x, r) \le c_0 r^m$, Δ_η be the normalized Laplacian. In this paper we prove that the semilinear heat equation $u_t = \Delta_\eta u + u^{1+\alpha}$ on G has no non-negative global solutions for any bounded, non-negative and non-trivial initial value in the case of $m\alpha = 2$. The obtained result provides a significant complement to the work that was done recently by Lin and Wu (2017) concerning the existence and nonexistence of global solutions for the semilinear heat equation on graphs.

1. Introduction and main results

In 1966, Fujita [8] studied the Cauchy problem of the semilinear heat equation

$$\begin{cases} u_t = \Delta u + u^{1+\alpha}, & (t, x) \in (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = a(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

and concluded that the solution is not global when $0 < N\alpha < 2$ and there is a global solution for a sufficiently small initial value when $N\alpha > 2$. The case $N\alpha = 2$ was investigated by Aronson and Weinberger [1], Hayakawa [9], Hu [11], Weissler [15] respectively. The conclusion is that when $N\alpha = 2$, Eq. (1.1) has no global solutions for any non-negative and non-trivial initial value.

Motivated by Fujita [8], the above-mentioned semilinear heat Eq. (1.1) was recently considered by Lin and Wu [13] on graphs, which can be regarded as a discrete analogue of the semilinear heat equation

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https://doi.org/10.1016/j.na.2018.01.012

on Euclidean space. Lin and Wu [13] discussed the existence and nonexistence of global solutions for the following semilinear heat equation on a locally finite connected weighted graph:

$$\begin{cases} u_t = \Delta u + u^{1+\alpha}, & (t,x) \in (0,+\infty) \times V, \\ u(0,x) = a(x), & x \in V, \end{cases} \tag{1.2}$$

where α is a positive parameter, $a(x)$ is bounded, non-negative and non-trivial in V.

Following the ideas of Coulhon and Grigor'yan [6] used for dealing with heat kernel estimates on Riemannian manifolds, Lin and Wu [12] established an on-diagonal lower estimate of continuous-time heat kernels on graphs. By means of some results on heat kernel estimates from [2,10,12], Lin and Wu [13] proved that, when Δ is a $\mu-$Laplacian, for a graph satisfying curvature condition $CDE'(n,0)$ and uniform volume growth $\frac{1}{c'}r^m \leq \mathcal{V}(x,r) \leq c'r^m$ ($c' \geq 1$), if $0 < m\alpha < 2$, then there does not exist a non-negative global solution u of (1.2) for any bounded, non-negative and non-trivial initial value and if $m\alpha > 2$, then the non-negative solution u of (1.2) is global in time, provided that the initial value is small enough.

Obviously, the results of [13] do not include the case of $m\alpha = 2$. It should be noted that, before the present work, it was not clear whether there exists a non-negative global solution u of (1.2) in the case of $m\alpha = 2$. In this paper we will discuss this question under the assumption that G is a finite graph and Δ is a normalized Laplacian Δ_η.

Our main results are as follows:

Theorem 1.1. *Let G be a simple, finite, connected, weighted graph satisfying curvature condition $CDE'(n,0)$ and polynomial volume growth $\mathcal{V}(x,r) \leq c_0 r^m$ for $x \in V$, $r \geq r_0$, where c_0, m, r_0 are positive constants. Suppose $a(x)$ is bounded, non-negative, non-trivial in V. If $m\alpha = 2$, then the following Eq. (1.3) has no non-negative global solution in $[0,+\infty)$ for any bounded, non-negative and non-trivial initial value.*

$$\begin{cases} u_t = \Delta_\eta u + u^{1+\alpha}, & (t,x) \in (0,+\infty) \times V, \\ u(0,x) = a(x), & x \in V, \end{cases} \tag{1.3}$$

where Δ_η is a normalized Laplacian.

Remark 1.2. Bauer et al. [2] showed that $\mathbb{Z}^m$ satisfies $CDE'(4.53m,0)$ for the normalized graph Laplacian. Moreover, the integer grid $\mathbb{Z}^m$ admits polynomial volume growth $\mathcal{V}(x,r) \leq c_0 r^m$. Thus, from Theorem 1.1 we obtain immediately the following corollary:

Corollary 1.3. *Let the graph G be $\mathbb{Z}_q^m$, where $\#V = q^m < \infty$. Suppose $a(x)$ is bounded, non-negative, non-trivial in V. If $m\alpha = 2$, then Eq. (1.3) has no non-negative global solution in $[0,+\infty)$ for any bounded, non-negative and non-trivial initial value.*

The remaining parts of this paper are organized as follows. In Section 2, we introduce some notations and definitions on graphs. In Section 3, we recall some known results about the heat kernel on graphs which are essential to prove the main results of this paper. In Section 4, we give the proof of Theorem 1.1. In Section 5, we provide a numerical experiment to demonstrate the assertion of Corollary 1.3.

2. Preliminaries

Let $G = (V,E)$ denote a simple, finite, connected graph. Here V denotes the set of vertices and E denotes the set of edges. Two vertices x and y are said to be adjacent if they are connected directly by an edge, in symbols $x \sim y$. In this paper we study weighted graphs, i.e., we allow the edges and vertices on G to be