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On the stochastic Cahn–Hilliard equation with a singular double-well potential☆

Luca Scarpa

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom

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A B S T R A C T

We prove well-posedness and regularity for the stochastic pure Cahn–Hilliard equation under homogeneous Neumann boundary conditions, with both additive and multiplicative Wiener noise. In contrast with great part of the literature, the double-well potential is treated as generally as possible, its convex part being associated to a multivalued maximal monotone graph everywhere defined on the real line on which no growth nor smoothness assumptions are assumed. The regularity result allows to give appropriate sense to the chemical potential and to write a natural variational formulation of the problem. The proofs are based on suitable monotonicity and compactness arguments in a generalized variational framework.

1. Introduction

The well-known Cahn–Hilliard equation was first introduced in [7] to describe the evolution of the phase separation phenomenon involving a binary metallic alloy: in general, in a solid–solid phase separation each phase concentrates, and this results in what is usually referred to as *spinodal decomposition*.

The classical Cahn–Hilliard equation reads

$$\partial_t u - \Delta w = 0\,, \quad w \in -\Delta u + \beta(u) + \pi(u) + g \qquad \text{in } (0,T) \times D,$$

where $D \subseteq \mathbb{R}^N$ $(N = 2, 3)$ is a smooth bounded domain with smooth boundary Γ, $T > 0$ is a fixed finite time, Δ stands for the Laplacian acting on the space variables and g is a given source. The unknown u and w represent the order parameter and the chemical potential, respectively. Here, β is the subdifferential of

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E-mail address: luca.scarpa.15@ucl.ac.uk.

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the convex part j and π is the derivative of the concave perturbation $\widehat{\pi}$ of a so-called double-well potential $\psi := j + \widehat{\pi}$. Typical examples of ψ (see also [16]) are given by

$$\psi_{reg}(r) = \frac{1}{4}(r^2-1)^2\,, \quad r \in \mathbb{R}\,,$$
$$\psi_{log}(r) = ((1+r)\ln(1+r) - (1-r)\ln(1-r)) - cr^2\,, \quad r \in (-1,1)\,, \quad c > 0\,,$$
$$\psi_{2obst}(r) = \begin{cases} c(1-r^2) & \text{if } |r| \leq 1\,, \\ +\infty & \text{if } |r| > 1\,, \end{cases}$$

which correspond to a regular, logarithmic and non-smooth double-well potential, respectively (the last one is usually considered in the so-called double-obstacle problem). In the simplest case, the equation is coupled with homogeneous Neumann boundary conditions for both u and w, and a given initial datum:

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = 0 \quad \text{on } (0,T)\times\Gamma\,, \qquad u(0) = u_0 \quad \text{in } D,$$

where the symbol $\mathbf{n}$ stands for the outward normal unit vector on Γ. It is well-known that the homogeneous Neumann condition for the chemical potential ensures the conservation of the mean-value of u on D, as it is easily proved integrating the first equation on D. It is also noteworthy that the term $w - g$ associated to the chemical potential results from the subdifferentiation of the Ginzburg–Landau free energy functional

$$\mathcal{E}(x) := \frac{1}{2}\int_D |\nabla x|^2 + \int_D (j(x) + \widehat{\pi}(x))\,.$$

From a mathematical perspective, deterministic Cahn–Hilliard equations have received much attention in the last years and have been analytically investigated also in more general frameworks, such as the viscous case and under the so-called dynamic boundary conditions. A special mention goes to the contributions [8–11,13,14,16,26] dealing with global well-posedness and regularity for Cahn–Hilliard and Allen–Cahn type equations with singular potentials, and [15,19,27] regarding asymptotics and long-time behaviour of solutions. Also, we point out the papers [12,17,18,30] concerning optimal control problems related to Cahn–Hilliard systems.

While the deterministic Cahn–Hilliard equation provides a good description of the spinodal decomposition process, on the other hand it is not effective in taking into account the effects due to the random solute vibrational movements. These can be accounted for directly by adding a cylindrical Wiener process W in the equation itself, hence getting a stochastic partial differential equation of the form

$$du(t) - \Delta w(t)\,dt = B(t,u(t))\,dW_t\,, \quad w \in -\Delta u + \beta(u) + \pi(u) + g \qquad \text{in } (0,T)\times D,$$

with homogeneous Neumann conditions for u and w, and a given initial value u_0, where B is a suitable stochastically integrable operator.

The available literature on stochastic Cahn–Hilliard equations is not as extended as the corresponding deterministic one and is mainly focused on the classical case of a smooth polynomial double-well potential. Let us point out the contribution [21], in which the authors prove existence and regularity of solutions, as well as existence and uniqueness of an invariant measure for the transition semigroup, in the case of a polynomial double-well potential of even degree $2p$. Moreover, in the case of the regular double-well potential ψ_{reg}, existence, uniqueness and regularity of weak statistical solution and existence of a strong solution are proved in [25] for the equation in local form and in [20] for a nonlocal version. Finally, let us also mention the contributions [1] on a stochastic Cahn–Hilliard equation with unbounded noise and [23,24,28] dealing with stochastic Cahn–Hilliard equations with reflections.

The noteworthy feature of this work is that neither growth nor smoothness assumptions on β are required, provided that β is everywhere defined: consequently, in contrast with great part of the existing literature, we are able to handle any double-well potential $\psi : \mathbb{R} \to \mathbb{R}$, not necessarily smooth, with any arbitrary