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The second expansion of the unique vanishing at infinity solution to a singular elliptic equation[☆]

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ABSTRACT

This paper is considered with the second expansion of the unique vanishing at infinity solution to $-\Delta u = b(x)g(u)$, $x \in \mathbb{R}^N (N \geq 3)$, where $b \in \mathrm{C}(\mathbb{R}^N)$ is nonnegative and may be supercritical attenuation or critical attenuation at infinity, $g \in \mathrm{C}^1((0,\infty),(0,\infty))$ is non-increasing on $(0,\infty)$ with $\lim_{s\to 0^+} g(s) = \infty$ and g is normalized regularly varying at zero with index $-\gamma$ ($\gamma > 1$).

1. Introduction and main results

In this article, we are interested in the second expansion of the unique vanishing at infinity solution to the following singular elliptic equation

$$-\Delta u = b(x)g(u), \tag{1.1}$$

where $x \in \mathbb{R}^N (N \geq 3)$, and a vanishing at infinity solution of Eq. (1.1) means that $u \in C^2(\mathbb{R}^N)$ solves Eq. (1.1) and $\lim_{|x|\to\infty} u(x) = 0$. The nonlinearity g satisfies

($\mathbf{g_1}$) $g \in C^1((0,\infty),(0,\infty))$ is non-increasing on $(0,\infty)$ and $\lim_{s\to 0^+} g(s) = \infty$;

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($\mathbf{g_2}$) there exist $\gamma > 1$ and a function $f \in C^1(0, a_1) \cap C[0, a_1)$ for $a_1 > 0$ small enough such that

$$\frac{-sg'(s)}{g(s)} := \gamma + f(s) \text{ with } \lim_{s\to 0^+} f(s) = 0,$$

i.e.,

$$g(s) = c_0 s^{-\gamma} \exp\left(\int_s^{a_1} \frac{f(\tau)}{\tau} d\tau\right),\ c_0 = g(a_1)a_1^\gamma,$$

where f satisfies one of the following conditions between

($\mathbf{S_1}$) $f \equiv 0$ on $(0, a_1]$;
($\mathbf{S_2}$) $f(s) \neq 0$, $\forall\, s \in (0, a]$ for some $a \leq a_1$.

Moreover, if ($\mathbf{S_2}$) holds in ($\mathbf{g_2}$), then we assume that

($\mathbf{g_3}$) there exists $\theta \geq 0$ such that

$$\lim_{s\to 0^+} \frac{sf'(s)}{f(s)} = \theta \geq 0.$$

If $\theta = 0$ in ($\mathbf{g_3}$), then we further assume that

($\mathbf{g_4}$) there exist $\beta > 0$ and $\sigma \in \mathbb{R}$ such that

$$\lim_{s\to 0^+} (-\ln s)^\beta f(s) = \sigma.$$

The weight b satisfies

($\mathbf{b_1}$) $b \in C(\mathbb{R}^N)$ is nonnegative in $\mathbb{R}^N$;
($\mathbf{b_2}$) there exist $k \in \mathcal{K}$, $\lambda \geq 2$ and $B_0 \in \mathbb{R}$ such that

$$b(x) = |x|^{-\lambda} k(|x|)\big(1 + B_0|x|^{-1} + o(|x|^{-1})\big),$$

where $\mathcal{K}$ denotes the set of Karamata functions k defined on $[s_0, \infty)$ by

$$k(s) := c \exp\left(\int_{s_0}^s \frac{y(\tau)}{\tau} d\tau\right),\ s \geq s_0 > 0$$

with $c > 0$ and $y \in C[s_0, \infty)$ such that $\lim_{s\to\infty} y(s) = 0$.

Eq. (1.1) arises in the study of boundary layer phenomena for viscous fluids, non-Newtonian fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials, and has been discussed extensively by many authors in different contexts.

If Ω is a bounded domain with C^2-boundary and $b \equiv 1$ in Ω, g satisfies ($\mathbf{g_1}$), then Fulks and Maybee [20], Stuart [44], Crandall, Rabinowitz and Tartar [13] proved that Eq. (1.1) possesses a unique vanishing at boundary solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$. In particular, the authors in [13] investigated the first estimate of the unique solution. If $f(u) = u^{-\gamma}$ with $\gamma > 0$, $b \in C^\alpha(\bar{\Omega})$, $b(x) > 0$ for all $x \in \bar{\Omega}$, then Lazer and McKenna [32] showed that Eq. (1.1) possesses a unique vanishing at boundary solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ and $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. If g satisfies ($\mathbf{g_1}$) and the conditions

($\mathbf{g_{01}}$) there exist positive constants c_0, η and $\gamma \in (0, 1)$ such that $g(s) \leq c_0 s^{-\gamma}$, for all $s \in (0, \eta_0)$;
($\mathbf{g_{02}}$) there exist constants $\theta > 0$ and $\eta_1 \geq 1$ such that $g(\xi s) \geq \xi^{-\theta} g(s)$ for all $\xi \in (0, 1)$ and $t \in (0, \xi\eta_1)$;
($\mathbf{g_{03}}$) the mapping $\xi \in (0, \infty) \to T(\xi) = \lim_{s\to 0^+} \frac{g(\xi s)}{\xi g(s)}$ is a continuous function,