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Weak Harnack estimates for supersolutions to doubly degenerate parabolic equations

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ABSTRACT

We establish weak Harnack inequalities for positive, weak supersolutions to certain doubly degenerate parabolic equations. The prototype of this kind of equations is

$$\partial_t u - \operatorname{div} |u|^{m-1}|Du|^{p-2}Du = 0, \quad p > 2, \quad m + p > 3.$$

Our proof is based on Caccioppoli inequalities, De Giorgi's estimates and Moser's iterative method.

1. Introduction

Harnack inequality for nonnegative solutions of the linear parabolic equations was established by Moser [24]. Furthermore, Aronson and Serrin [1], Trudinger [27] and Ivanov [12] independently extended Moser's result to the quasilinear case. Later DiBenedetto [4] established intrinsic Harnack inequality for weak solutions to the parabolic p-Laplace equations. The proof is based on the idea of time-intrinsic geometry, maximum principle and comparison to Barenblatt solutions. Recently, DiBenedetto, Gianazza and Vespri [6,7] extended this result to the parabolic equations with general quasi-linear structure.

The issue of the Harnack inequality for weak supersolutions to quasi-linear parabolic equations was settled by Trudinger [27]. However, the weak supersolutions may not be continuous and they do not, in general, satisfy an intrinsic Harnack inequality (see for instance [23]). In [19], Kuusi proved that any nonnegative weak

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supersolutions to a certain evolutionary p-Laplace equations satisfy an integral form of the intrinsic Harnack inequality. Similar result for the degenerate porous medium equations was announced by DiBenedetto, Gianazza and Vespri [6] and later proved by Lehtelä [22]. This kind of weak Harnack inequality was used by Kuusi, Lindqvist and Parviainen [21] to the study of summability of unbounded supersolutions.

In this work, we are interested in the doubly degenerate parabolic equations whose prototype is

$$\frac{\partial u}{\partial t} - \operatorname{div} |u|^{m-1}|Du|^{p-2}Du = 0. \tag{1.1}$$

This equation models the filtration of a polytropic non-Newtonian fluid in a porous medium (see for example [18]). Porzio and Vespri [26], and Ivanov [13] independently proved that the weak solutions to (1.1) are Hölder continuous. The Harnack inequality of weak solutions to doubly degenerate parabolic equations has been established by Vespri [28] and the case of equations with general quasi-linear structure was treated by Fornaro and Sosio [9]. Motivated by these work, we are interested in finding weak Harnack inequalities for positive, weak supersolutions to this kind of parabolic equations.

The aim of this paper is to establish both local and global Harnack estimates for positive, weak supersolutions to (1.1) in the range $p + m > 3$ and $p > 2$. Our proof is in the spirit of [7, chapter 3] which reduces the proof to the consideration of hot alternative and cold alternative. Our first goal is to prove the Caccioppoli estimates. The treatment of the weak supersolutions is different from weak solutions, since the test function should be nonnegative. Unlike the argument in [9], it is not convenient to use Steklov average in the context of supersolutions. Instead, we use the time mollification introduced by Naumann [25]. This kind of the time mollification was also used by Kinnunen and Lindqvist [15–17] to establish the priori estimates of supersolutions. The major difficulty in the proof of Caccioppoli estimates for the cold alternative stems from the fact that any supersolution to (1.1) plus a constant may not be a supersolution anymore. We have to use the dampening function introduced by Lehtelä [22] to construct a suitable test function in the proof. This idea has also been used by Ivert, Marola and Masson [14] in a different context. Subsequently, we establish a result concerning expansion of positivity and iterate this result to obtain an estimate for the supersolutions in hot alternative. Furthermore, we have to work with the cold alternative invented by Kuusi [19, section 5]; see also [22, section 4]. In order to perform a Moser's iterative argument for the cold alternative, an improvement for the parabolic Sobolev embedding estimates is needed.

An outline of this paper is as follows. We set up notations and state the main results in Section 2. In Section 3, we prove some parabolic Sobolev inequalities which will be used in Section 7. Subsequently, Section 4 establishes the Caccioppoli estimates, which play a crucial role in the remainder of the proof. Section 5 contains a discussion of the expansion of positivity, while in Section 6 we prove an estimate from below for the supersolutions in the hot alternative. In Section 7, we use Moser's iterative method to obtain a lower bound for the supersolution in the cold alternative. Finally, in Section 8, we finish the proof of the local and global weak Harnack inequalities.

2. Statement of the main results

Throughout the paper, E will denote a bounded domain in $\mathbb{R}^N$ and ∂E stand for the boundary of E. For $T > 0$, let E_T be the cylindrical domain $E \times (0, T]$. Points in $\mathbb{R}^{N+1}$ will be denoted by $z = (x, t)$ where $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

For $f \in C^1(E_T)$, we denote by Df the differentiation with respect to the space variables, while $\partial_t f$ stands for the time derivative. The spaces $L^p(E)$ and $W^{1,p}(E)$ are the locally Lebesgue and Sobolev spaces. A function $f \in L^p_{\text{loc}}(E)$ if $f \in L^p(K)$ for all compact subset $K \subset E$. Similarly, the function $f \in W^{1,p}_{\text{loc}}(E)$ if $f \in W^{1,p}(K)$ for all compact subset $K \subset E$.