Nonlinear Analysis 170 (2018) 123–141

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Stabilization in the logarithmic Keller–Segel system

Michael Winkler[a], Tomomi Yokota[b,*]

[a] Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany
[b] Department of Mathematics, Tokyo University of Science, Tokyo 162-8601, Japan

ARTICLE INFO

Article history:
Received 17 October 2017
Accepted 4 January 2018
Communicated by S. Carl

MSC:
primary 35B40
secondary 35K65
92C17

Keywords:
Chemotaxis
Singular sensitivity
Large time behavior

ABSTRACT

The Keller–Segel system

$$\begin{cases} u_t = D\Delta u - D\chi\nabla\cdot\left(\frac{u}{v}\nabla v\right), & x\in\Omega,\ t>0,\\ v_t = D\Delta v - v + u, & x\in\Omega,\ t>0,\end{cases}$$

is considered in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$, with smooth boundary, where $\chi>0$ and $D>0$. The main results identify a condition on the parameters $\chi<\sqrt{\frac{2}{n}}$ and $D>0$, essentially reducing to the assumption that $\frac{\chi^2}{D}$ be suitably small, under which for all reasonably regular and positive initial data the corresponding classical solution of an associated Neumann initial–boundary value problem, known to exist globally according to previous findings, approaches the homogeneous steady state $(\overline{u}_0,\overline{u}_0)$ at an exponential rate with respect to the norm in $(L^\infty(\Omega))^2$ as $t\to\infty$, where $\overline{u}_0 := \frac{1}{|\Omega|}\int_\Omega u(\cdot,0)$. As a particular consequence, this entails a convergence statement of the above flavor in the normalized system with $D=1$ and fixed $\chi<\sqrt{\frac{2}{n}}$, provided that Ω satisfies a certain smallness condition.

1. Introduction

Among the rich variety of Keller–Segel-type models discussed in the literature, the problem

$$\begin{cases} U_t = \Delta U - \chi\nabla\cdot\left(\frac{U}{V}\nabla V\right), & y\in\Omega_1,\ t>0,\\ V_t = \Delta V - V + U, & y\in\Omega_1,\ t>0,\\ \frac{\partial U}{\partial\nu} = \frac{\partial V}{\partial\nu} = 0, & y\in\partial\Omega_1,\ t>0,\\ U(y,0)=U_0(y),\quad V(y,0)=V_0(y), & y\in\Omega_1,\end{cases}\tag{1.1}$$

with $\chi>0$ and given nonnegative functions U_0 and V_0 appears to be of particular interest. This on the one hand reflects that in refinement of the classical minimal Keller–Segel system [19] in which the evolution of the

* Corresponding author.
E-mail addresses: michael.winkler@math.uni-paderborn.de (M. Winkler), yokota@rs.kagu.tus.ac.jp (T. Yokota).

https://doi.org/10.1016/j.na.2018.01.002
0362-546X/

cell density U is governed by $U_t = \Delta U - \nabla \cdot (U \nabla S(V))$ with linear S, the choice of the *logarithmic sensitivity* $S(V) = \chi \ln V$ in (1.1) with regard to the chemical signal concentration V is well-adapted to situations in which the chemotactic response of cells respects the long-familiar Weber–Fechner law of stimulus perception ([27,20]). On the other hand, this modification goes along with significant mathematical challenges which are inter alia due to an apparent loss of a treasured gradient structure that has been underlying essential bodies of the analysis in the case of linear sensitivities (see e.g. [26,16,8,33] and also the survey [2]).

As a consequence, it seems yet widely unknown to which extent the decay of the derivative $S'(V) = \frac{\chi}{V}$ at large signal densities may suppress phenomena of blow-up which constitute the probably most characteristic qualitative feature of original Keller–Segel models with $S' \equiv const.$ in two- or higher-dimensional spatial domains [15,33]. After all, partial results indicate a substantial dampening effect at least for small values of the factor χ in (1.1), asserting global existence of bounded classical solutions for widely arbitrary positive initial data, and thus ruling out explosions, when Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $\chi < \chi_0(n)$ with some $\chi_0(n) > 0$ which is currently known to satisfy $\chi_0(2) > 1.015$ and $\chi_0(n) \geq \sqrt{\frac{2}{n}}$ for $n \geq 3$ [21,3,32,10]; cf. also [35,24]. For larger values of χ, nontrivial global solutions have been constructed only in generalized frameworks so far, thus yet allowing solutions to become unbounded even within finite time, but at least excluding any collapse into persistent Dirac-type singularities. For instance, if $\chi < \sqrt{\frac{n+2}{3n-4}}$ then (1.1) is solvable already within quite a natural weak solution concept [32], while under the weaker assumption that $\chi < \sqrt{\frac{n}{n-2}}$, in radially symmetric settings global solutions are known to exist in a slightly more generalized framework [29]. Only recently, within a yet weaker solution concept this assumption could be further relaxed in the sense that merely requiring

$$\chi < \begin{cases} \infty & \text{if } n = 2, \\ \sqrt{8} & \text{if } n = 3, \\ \dfrac{n}{n-2} & \text{if } n \geq 4, \end{cases} \tag{1.2}$$

is sufficient to allow for corresponding global solvability, even without any symmetry assumption [22]. We remark that for parabolic–elliptic simplifications of (1.1) in which the second equation therein is replaced with $0 = \Delta V - V + U$, somewhat more comprehensive results are available, but beyond this furthermore providing inter alia providing some examples of exploding solutions when $n \geq 3$ and $\chi > \frac{2n}{n-2}$ [25], extending the range of χ in (1.2) for generalized solvability when $n = 3$ [4], and especially in the two-dimensional case asserting global classical solvability regardless of the size of χ [12]; a result of the latter flavor could recently even be carried over to the radial version of the fully parabolic variant of (1.1) with its second equation becoming $\tau V_t = \Delta V - V + U$ for suitably small $\tau > 0$ [13].

Beyond these fundamental results on global solvability, however, only little seems known about solutions to (1.1); in particular, their qualitative behavior seems widely unaddressed in the literature, with available exceptions concentrating on the associated steady state problem. Indeed, a large variety of highly nontrivial equilibria have been detected since the seminal work [23] in this direction [6,1,7,5], but their role, as actually the role of any stationary solution, in the dynamics of the parabolic problem (1.1) seems unclear up to now.

Main results. The intention of this paper is to accept the challenge of describing the large time behavior in systems of the form (1.1) despite lacking knowledge on any meaningful global gradient flow structure therein. In order to achieve nontrivial progress in this direction going beyond straightforward perturbation arguments leading no further than to local stability and attractivity results, unlike the very few precedent energy-independent cases of large-time analysis in chemotaxis problems which exclusively seem to rely on the availability of comparison arguments (see e.g. [28,36,30,34]), we shall pursue a strategy heuristically motivated by the trivial observation that both the numerator and the denominator in the quotient $\frac{U}{V}$ in (1.1) grow linearly with respect to U and V, respectively. Therefore, namely, naively assuming that up to parabolic smoothing V will become conveniently large wherever U attains large values and vice versa, our