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Multiplicity of concentrating positive solutions for Schrödinger–Poisson equations with critical growth

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ABSTRACT

In this paper we deal with a nonlinear Schrödinger–Poisson equation involving critical Sobolev exponent. Under a local condition imposed on the potential, we relate the number of positive solutions with the topology of the set where the potential attains its minimum values. In the proofs we apply Szulkin and Weth's generalized Nehari manifold methods, penalization techniques and Ljusternik–Schnirelmann theory.

1. Introduction

In recent years, the following nonlinear Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

have been widely investigated and it is well known that they have a strong physical meaning because they appear in quantum mechanics models (see e.g. [13,26]) and in semiconductor theory [28,30]. In particular, systems like (1.1) have been introduced in [11,12] as a model describing solitary waves, for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field, and are usually known as Schrödinger–Poisson systems. We refer to [11] for more details on physical aspects.

Many researches have been devoted to the study of (1.1) in the recent literature, see for example, [5,6,7,15,23,28,34,40,43] and the references therein. In [5], Ambrosetti and Ruiz obtained multiple

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solutions for (1.1) by minimax methods combined with the monotonicity skills. D'Aprile and Mugnai [6] proved the existence of infinitely many radial symmetric solutions for a Schrödinger–Poisson equation (1.1) with $K(x) \equiv 1$. Cerami and Vaira [15] proved the existence of positive solutions for (1.1) with $V(x) \equiv 1, f(x,u) = a(x)|u|^{p-1}u, p \in (3,5)$, not requiring any symmetry property on the potential $a(x)$. In [9], Azzollini and Pomponio proved the existence of a ground state solution for (1.1) with $K(x) \equiv 1, f(x,u) = |u|^{p-1}u$ for $2 < p < 5$. Ruiz [28] studied the following Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + u + \lambda\phi u = u^p, \\ -\Delta\phi = u^2, \quad \lim\limits_{|x|\to\infty} \phi(x) = 0. \end{cases} \tag{1.2}$$

The author gives existence and nonexistence results, depending on the parameters p and λ.

For the single parameter-perturbed Schrödinger–Poisson equation

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi(x)u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

D'Aprile and Wei [8] constructed positive radially symmetric bound states of (1.3); Ruiz and Vaira [35] proved the existence of multi-bump solutions of (1.3), whose bumps concentrate around a local minimum of the potential $V(x)$ when $f(u) = u^p, 3 < p < 5$. Ambrosetti in [3] considered (1.3) with $f(u) = |u|^{p-1}u$ and proved that (1.3) has a positive solution by using perturbation methods [4]. Ianni and Vaira [24] obtained a single-bump solution of (1.3) near critical points of $V(x)$. In [33], Ruiz proved that the semiclassical states of (1.3) can concentrate around a sphere when V satisfies suitable conditions.

When $\phi \equiv 0$, Eq. (1.3) reduces to the following well-known Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(u), \tag{1.4}$$

which has been extensively studied in the past few decades, and there are many existence and concentration results in the past decades. We refer to [4,1,2,39] and the references therein.

In this paper we consider the following double parameters perturbed Schrödinger–Poisson equation with critical growth

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u) + |u|^4u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3),\ u(x) > 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SP}_\varepsilon$}$$

where $\varepsilon > 0$ is a parameter, the potential $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy some conditions which will be stated later on. In [20] the author studied the multiplicity of positive solutions to $(\mathcal{SP}_\varepsilon)$ without the critical term $|u|^4u$, by using Ljusternik–Schnirelmann theory and minimax methods. The nonlinearity f is assumed to be C^1 function and satisfy the following differentiability condition:

(D) There exist constants $\sigma \in (4,6), Q > 0$ such that $f'(t)t^2 - 3f(t)t \geq Qt^\sigma$ for all $t \geq 0$.

Later, He and Zou [21] considered the existence and concentration behavior of positive solutions for $(\mathcal{SP}_\varepsilon)$ by variational methods. However, in both papers [18,19], the nonlinearity f is assumed to be C^1 class and the potential V needs to verify a global condition introduced by Rabinowitz [32], namely,

(R) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{\mathbb{R}^3} V(x) > 0$.

Recently, M. Yang [42] studied the existence of positive solution to $(\mathcal{SP}_\varepsilon)$ with competing potentials by using variational method. J. Wang et al. [38] investigated the existence and concentration of positive solutions to problem $(\mathcal{SP}_\varepsilon)$ with subcritical growth. Y. He and G. Li [22] showed the existence of positive