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On restrictions of Besov functions

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ABSTRACT

In this paper, we study the smoothness of restrictions of Besov functions. It is known that for any $f \in B^s_{p,q}(\mathbb{R}^N)$ with $q \leq p$ we have $f(\cdot, y) \in B^s_{p,q}(\mathbb{R}^d)$ for a.e. $y \in \mathbb{R}^{N-d}$. We prove that this is no longer true when $p < q$. Namely, we construct a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that $f(\cdot, y) \notin B^s_{p,q}(\mathbb{R}^d)$ for a.e. $y \in \mathbb{R}^{N-d}$. We show that, in fact, $f(\cdot, y)$ belong to $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$ for a.e. $y \in \mathbb{R}^{N-d}$, a Besov space of generalized smoothness, and, when $q = \infty$, we find the optimal condition on the function Ψ for this to hold. The natural generalization of these results to Besov spaces of generalized smoothness is also investigated.

1. Introduction

In this paper, we address the following question: given a function $f \in B^s_{p,q}(\mathbb{R}^N)$,

what can be said about the smoothness of $f(\cdot, y)$ for a.e. $y \in \mathbb{R}^{N-d}$?

In order to formulate this as a meaningful question, one is naturally led to restrict oneself to $1 \leqslant d < N$, $0 < p, q \leqslant \infty$ and $s > \sigma_p$, where

$$\sigma_p = N\left(\frac{1}{p} - 1\right)_+, \tag{1.1}$$

since otherwise $f \in B^s_{p,q}(\mathbb{R}^N)$ need not be a regular distribution.

Let us begin with a simple observation. If $f \in L^p(\mathbb{R}^N)$ for some $0 < p \leqslant \infty$, then

$$f(\cdot, y) \in L^p(\mathbb{R}^d) \qquad \text{for a.e.} \quad y \in \mathbb{R}^{N-d}.$$

This is a straightforward consequence of Fubini's theorem. Using similar Fubini-type arguments, one can show that, if $f \in W^{s,p}(\mathbb{R}^N)$ for some $0 < p \leqslant \infty$ and $\sigma_p < s \notin \mathbb{N}$, then we have $f(\cdot, y) \in W^{s,p}(\mathbb{R}^d)$ for a.e. $y \in \mathbb{R}^{N-d}$. We say that these spaces have the *restriction property*.

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Unlike their cousins, the Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^N)$, Besov spaces do not enjoy the Fubini property unless $p = q$, that is

$$\sum_{j=1}^{N} \Big\| \|f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N)\|_{B^s_{p,q}(\mathbb{R})} \Big\|_{L^p(\mathbb{R}^{N-1})},$$

is an equivalent quasi-norm on $B^s_{p,q}(\mathbb{R}^N)$ if, and only if, $p = q$; while the counterpart for $F^s_{p,q}(\mathbb{R}^N)$ holds for any given values of p and q where it makes sense (see [42, Theorem 4.4, p.36] for a proof). In particular, $B^s_{p,p}(\mathbb{R}^N)$ and $F^s_{p,q}(\mathbb{R}^N)$ have the restriction property. It is natural to ask whether or not this feature holds in $B^s_{p,q}(\mathbb{R}^N)$ for an arbitrary $q \neq p$.

Let us recall some known facts.

Fact 1.1. *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < q \leqslant p \leqslant \infty$, $s > \sigma_p$ and $f \in B^s_{p,q}(\mathbb{R}^N)$. Then,*

$$f(\cdot, y) \in B^s_{p,q}(\mathbb{R}^d) \qquad \textit{for a.e.} \quad y \in \mathbb{R}^{N-d}.$$

(A proof of a slightly more general result will be given in the sequel, see Proposition 5.1.)

In fact, there is a weaker version of Fact 1.1, which shows that this stays "almost" true when $p < q$. This can be stated as follows

Fact 1.2. *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$, $s > \sigma_p$ and $f \in B^s_{p,q}(\mathbb{R}^N)$. Then,*

$$f(\cdot, y) \in \bigcap_{s' < s} B^{s'}_{p,q}(\mathbb{R}^d) \qquad \textit{for a.e.} \quad y \in \mathbb{R}^{N-d}.$$

See e.g. [22, Theorem 1] or [5, Theorem 1.1].

Mironescu [33] suggested that it might be possible to construct a counterexample to Fact 1.1 when $p < q$. We prove that this is indeed the case. This is quite remarkable since, to our knowledge, the list of properties of the spaces $B^s_{p,q}$ where q plays a crucial role is rather short.

Our first result is the following

Theorem 1.3. *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and let $s > \sigma_p$. Then, there exists a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, y) \notin B^s_{p,\infty}(\mathbb{R}^d) \qquad \textit{for a.e.} \quad y \in \mathbb{R}^{N-d}.$$

Note that this is actually stronger than what we initially asked for, since $B^s_{p,q} \hookrightarrow B^s_{p,\infty}$.

Remark 1.4. We were informed that, concomitant to our work, a version of Theorem 1.3 for $N = 2$ and $p \geqslant 1$ was proved by Mironescu, Russ and Sire in [34]. We present another proof independent of it with different techniques. In fact, we will even prove a generalized version of Theorem 1.3 that incorporates other related function spaces (see Theorem 6.1) which is of independent interest.

Despite the negative conclusion of Theorem 1.3, one may ask if something weaker than Fact 1.1 still holds when $p < q$. For example, by standard embeddings, we know that

$$B^s_{p,q}(\mathbb{R}^N) \hookrightarrow A^{s,p}(\mathbb{R}^N) \qquad \text{for any} \quad 0 < q < \infty,$$

where $A^{s,p}(\mathbb{R}^N)$ stands for respectively

$$C^{s-\frac{N}{p}}(\mathbb{R}^N), \ \ \text{BMO}(\mathbb{R}^N) \ \text{and} \ L^{\frac{Np}{N-sp},\infty}(\mathbb{R}^N), \tag{1.2}$$