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Harnack's inequality and Green's functions on locally finite graphs

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ABSTRACT

In this paper, we study the gradient estimate for positive solutions of Schrodinger equations on locally finite and connected graphs. Then we derive Harnack's inequality for positive solutions of the Schrodinger equations on such graphs. We also set up some existence results about Green's functions of the Laplacian equations on locally finite graphs. We derive a lower bound for the principal eigenvalue of the Laplace operator in terms of the upper bound of total integral of Green's function. Interesting existence results for positive solutions of Schrodinger equations are derived via the use of related principal eigenvalues.

1. Introduction

The main purpose of this paper is to obtain Green's functions and Harnack inequalities for Schrodinger equations on locally finite and connected graphs with weights. Green's functions and Harnack inequalities for Schrodinger equations on Riemannian manifolds are important objects in Geometric Analysis [8]. However, their counterparts in infinite graphs are at least nontrivial as one may see these aspects in the works [4,5]. In principle, one may work with analytical problems by use of curvature on graphs. However, one may not define the concept of curvature on discrete graphs as in Riemannian geometry. Though one can expect that the behavior of (Ricci) curvature plays a role, one needs to find new tools to obtain Green's functions and Harnack inequalities. These are what we have tried in this paper. We now consider some related references to our work here. In the interesting work [3], Yau and his coauthors studied the Harnack estimate for eigenfunctions defined by the Laplacian operator on the locally finite connected graph with non-negative Ricci curvature as they defined. In [1], Yau and his coauthors studied the Li–Yau type Harnack inequality for positive solutions and heat kernels to the heat equation defined on the locally finite connected graphs.

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The existence of heat kernels on such graphs had been done in the works [13,15,16]. For more results about heat equations on graphs, we refer to the recent papers [1,9].

Just like Poisson equation on graphs, the Schrodinger equations on graphs are of fundamental importance. Such equations arise naturally from the discrete process of their continuous counterparts. The discrete equations on graphs are also useful for numerical purpose. It is an interesting topic to understand the solution structure for linear Schrodinger equation on graphs. Harnack inequality for elliptic equations is one of the main tools for dealing with such problems (see Peter Li's book [8] for continuous case). In first part of this paper, we study the Harnack inequalities for positive solutions of a class of Schrodinger equations on locally finite connected graphs. We derive local Harnack inequalities for positive solutions to Schrodinger equations based on an improved gradient estimate in [12] (see also [10] for more related works). We use this chance to point out that Kato's inequalities had been previously found in [7]. Kato's inequalities have been used by us to study the Ginzburg–Landau equation [12] (see also [11] for more background). We use Kato's inequality and the maximum principle to understand the principal eigenvalues of Schrodinger equations. Just as we have known for elliptic partial differential equations of second order, the Harnack inequality for Schrodinger equations is a basic tool to obtain existence and compactness results for their solutions. The Harnack inequality on some special graphs has been obtained in [4].

Green's functions had been introduced in a famous essay by George Green in 1828. Green's functions have been extensively used in solving differential equations from mathematical physics. In particular, Green's functions are very useful to solve the Poisson equations. The concept of Green's functions has also had a pervasive influence in numerous areas. As pointed out in [5], Green's functions provide a powerful tool in dealing with a wide range of combinatorial problems. Many formulations of Green's functions occur in a variety of topics. There are some interesting works about Green's functions on graphs [2]. Based on our Harnack inequality, we consider the existence of global Green's functions for discrete Laplace equations defined on locally finite connected graphs. Although we use different tools, we can get results in graphs which have their counterparts in Riemannian geometry [8].

The plan of the paper is below. Notations and Harnack inequalities for positive solutions to Schrodinger equations are introduced in Section 2. All results about Green's functions are stated and proved in Section 3.

2. Gradient estimate and Harnack's inequality

We first recall some definitions and results from the book [2] and from our paper [12].

Let $(X, \mathcal{E})$ be a graph with countable vertex set X and edge set $\mathcal{E}$. We assume that the graph is *simple*, i.e., no loops and no multi-edges. We also assume that the graph is connected and each vertex has finite neighbors. We simply call the graphs with these properties *the locally finite connected graphs*. Let $\mu_{xy} = \mu_{yx} > 0$ be a symmetric weight on $\mathcal{E}$. We define

$$d_x = \sum_{(x,y)\in\mathcal{E}} \mu_{xy}$$

and call d_x the *degree of* $x \in X$. Since each vertex x has only finite neighbors, i.e.,

$$\sharp\{y \in X | (x,y) \in \mathcal{E}\} < \infty$$

for each $x \in X$, we know that $d_x < \infty$ for all $x \in X$. It is also clear that the number

$$\hat{d}_x = \sup_{(x,y)\in\mathcal{E}} \frac{d_x}{\mu_{xy}}$$

is also finite for each $x \in X$ and this number is also important in our gradient estimate below. We use $d(x,y)$ to denote the distance between the vertices x and y in X. Sometimes, we write $x \sim y$ for the relation $(x,y) \in \mathcal{E}$.