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Multiple critical points of saddle geometry functionals

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ABSTRACT

We study the multiplicity of critical points for continuously differentiable functionals on real Banach spaces. We prove that a functional which satisfies the assumptions of the Saddle Point Theorem and moreover is bounded from below has at least three critical points. Apparently, there is a global minimizer and a saddle point and we show the existence of a third critical point. The idea of the proof is based on the minus-gradient flow. This result is closely related to the three critical points theorem of H. Brezis and L. Nirenberg which assumes a local linking. Finally, we apply the result on the Dirichlet problem for semilinear stationary PDEs. The analysis includes, for example, the existence of multiple stationary solutions of bistable (or Allen–Cahn) equation and semipositone problems.

1. Introduction and main result

The critical point theory has proved to be a very important and efficient tool in mathematics. It readily became one of fundamental building blocks of nonlinear analysis. Especially, the connection with the existence theory for differential equations has attracted a lot of attention. Now it is a standard way of proving miscellaneous results such as bifurcation theorems, mentioned existence results for differential equations, stability results in the theory of dynamical systems and many others (see, e.g., J. Mawhin, M. Willem [24]).

We study the multiplicity of critical points for continuously differentiable functionals on real Banach spaces which has certain saddle-type geometry. In 1978 P.H. Rabinowitz [31] introduced the Saddle Point Theorem which became quickly an essential part of the critical point theory and one of principal minimax theorems. It relies on a special geometrical assumption which naturally abstracts the idea of saddle points in finite dimension. In this paper we show that if the considered functional satisfies the P.H. Rabinowitz's saddle

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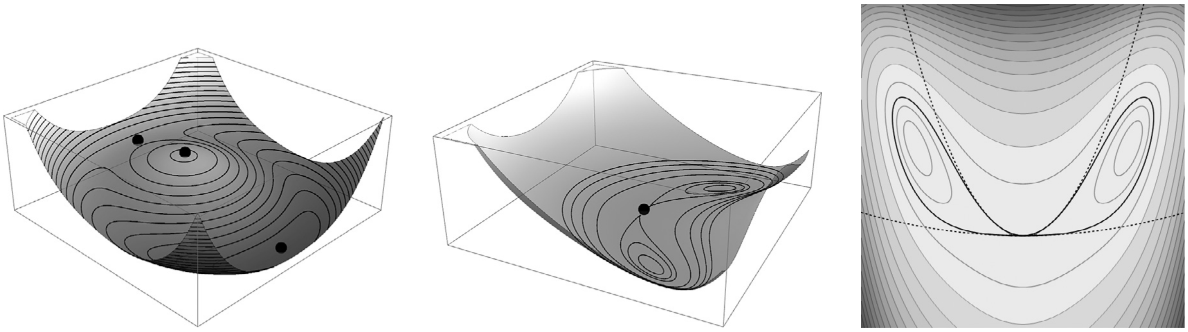

(a) The graph of function $F(\cdot,\cdot,0)$ for F given by (1.1) with its critical points.

(b) The graph of function F given by (1.2) illustrating a saddle which is not covered by assumption (BN).

(c) Contours of function F given by (1.2) with the zero contour being highlighted and squeezed by two parabolas, see (1.3).

Fig. 1.

assumption (denoted below by (R)) together with the Palais–Smale compactness condition and additionally is bounded from below, it has at least three critical points.

The following theorem is our main result. We denote by $B_Y(R)$ the open ball with the radius $R > 0$ in the subspace $Y \subset X$ centered at the origin, by $\partial B_Y(R)$ the corresponding sphere in Y and by X^* the dual space of X.

Theorem 1.1. *Let X be a real Banach space, $X = Y \oplus Z$ where $Y \neq \{o\}$ is finite dimensional. Assume that $F \in C^1(X, \mathbb{R})$, is bounded from below and satisfies*

(R) *there exists $R > 0$ such that $\max_{u\in\partial B_Y(R)} F(u) < \inf_{u\in Z} F(u)$,*

and the Palais–Smale condition

(PS) *every sequence $\{u_n\} \subset X$ such that $\{F(u_n)\} \subset \mathbb{R}$ is bounded and $\|F'(u_n)\|_{X^*} \to 0$ possesses a convergent subsequence.*

Then F has at least three critical points.

The existence of two critical points in Theorem 1.1, specifically the global minimizer and a saddle point, follows immediately from one of minimization principles and from the Saddle Point Theorem. More ambitious problem is to show the presence of a third critical point.

Theorem 1.1 uses the splitting $X = Y \oplus Z$ where Y is a finite dimensional subspace. For $\dim Y = 1$ there exists a mountain range of F formed along the subspace Z separating X into two parts. Consequently, there is a minimizer on each side of that mountain range, since we assume that the functional is bounded from below. However, the subspace Z does not have the separation property provided $\dim Y > 1$. Hence, there need not be a second minimizer in this case.

As an example consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F(x, y, z) = \left(x^2 + y^2\right)\left(x^2 + y^2 - 1\right) + \frac{y}{4} + z^2. \tag{1.1}$$

One can easily show that the function F satisfies assumptions of Theorem 1.1 with $Y = \{(x, y, 0)\}$ and $Z = \{(0, 0, z)\}$ for which $\dim Y = 2$. Nonetheless, the function F has a unique minimizer. More precisely, F has the global minimizer and two saddle points (see Fig. 1(a)).