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Elliptic equations involving the 1-Laplacian and a subcritical source term

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ABSTRACT

In this paper we deal with a Dirichlet problem for an elliptic equation involving the 1-Laplacian operator and a source term. We prove that, when the growth of the source is subcritical, there exist two bounded nontrivial solutions to our problem. Moreover, a Pohožaev type identity is proved, which holds even when the growth is supercritical. We also show explicit examples of our results.

1. Introduction

This paper is concerned with the following Dirichlet problem for the 1-Laplacian operator and a subcritical source term, whose model problem is

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |u|^{q-1}u, & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded set with Lipschitz boundary and $0 < q < \frac{1}{N-1}$. Our aim is to obtain nontrivial solutions (in the sense of Definition 2.1) and study their properties.

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We point out that similar problems have many applications and have been studied for a long time. Indeed, steady states of reaction–diffusion equations have systematically been studied since the late 1970s (see [17,21] for a more recent survey). More precisely, Dirichlet problems with p-Laplacian type operator ($p > 1$) having a term with a subcritical growth, that is:

$$\begin{cases} -\Delta_p u = |u|^{q-1}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

with $0 < q < p^* - 1$ (where p^* stands for the Sobolev conjugate), have extensively been considered in the theory of Partial Differential Equations by using different approaches (for a background we refer to [2,16]). For instance in [15] the authors, by using the well-known "Mountain Pass Theorem" by Ambrosetti and Rabinowitz [3], firstly proved that the trivial solution is a local minimum of the corresponding energy functional and then, since the functional has a mountain pass geometry, they find other critical points (one positive and another one negative), which obviously are solutions to problem (2). We point out that the proof of the Palais–Smale condition relies on the reflexivity of the energy space $W_0^{1,p}(\Omega)$. Moreover, the restriction $q < p^* - 1$ ensures that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, being this fact essential for the approach used in [15].

The 1-Laplace operator appearing in (1) introduces some extra difficulties and special features. We recall that in recent years there have been many works devoted to this operator (we refer to the pioneering works [5,12,18,19] and the related papers [6,7,10,11,13,14]). One of the main interests for studying the Dirichlet problem for equations involving the 1- Laplacian comes from the variational approach to image restoration (we refer to [8] for a review on the first variational models in image processing and their connection with the 1-Laplacian). This has led to a great amount of papers dealing with problems that involve the 1-Laplacian operator. In spite of this situation, up to our knowledge, this is the first attempt to analyze problem (1).

The natural energy space to study problems involving the 1- Laplacian is the space $BV(\Omega)$ of functions of bounded variation, i.e., those L^1-functions such that their distributional gradient is a Radon measure having finite total variation. In order to deal with the 1-Laplacian operator, a first difficulty occurs by defining the quotient $\frac{Du}{|Du|}$, being Du just a Radon measure. It can be overcome through the theory of pairings of L^∞-divergence-measure vector fields and the gradient of a BV-function (see [9]). Using this theory, we may consider a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \le 1$ and $(\mathbf{z}, Du) = |Du|$, so that $\mathbf{z}$ plays the role of the above ratio. In general, the Dirichlet boundary condition is not achieved in the usual trace form, so that a very weak formulation must be introduced: $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$, where $[\mathbf{z}, \nu]$ stands for the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$.

We point out that the space $BV(\Omega)$ is not reflexive, so that we cannot follow the arguments of [15]. Instead, we apply the results in [15] for problem (2) getting nontrivial solutions w_p and then we let p goes to 1. Hence, one of our biggest concerns will be that constants appearing in the proof do not depend on p. The other major difficulty we have to overcome is to check that the limit function $w = \lim_{p\to 1} w_p$ is not trivial.

1.1. Assumptions and main result

Let us state our problem and assumptions more precisely. We consider the general problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = f(x,u), & \text{in } \Omega, \\ \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here, the source term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following hypotheses