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Existence of solutions to higher order Lane–Emden type systems

Delia Schiera

Università degli Studi dell'Insubria, Dipartimento di Scienza e Alta Tecnologia, via Valleggio 11, 22100 Como, Italy

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A B S T R A C T

We prove existence results for the Lane–Emden type system

$$\begin{cases} (-\Delta)^\alpha u = |v|^q & \text{in } B_1 \subset \mathbb{R}^N \\ (-\Delta)^\beta v = |u|^p & \\ \dfrac{\partial^r u}{\partial \nu^r} = 0, \ r = 0, \ldots, \alpha - 1, \ \text{on } \partial B_1 \\ \dfrac{\partial^r v}{\partial \nu^r} = 0, \ r = 0, \ldots, \beta - 1, \ \text{on } \partial B_1. \end{cases}$$

where B_1 is the unitary ball in $\mathbb{R}^N$, $N > \max\{2\alpha, 2\beta\}$, ν is the outward pointing normal, $\alpha, \beta \in \mathbb{N}$, $\alpha, \beta \geq 1$ and $(-\Delta)^\alpha = -\Delta((-\Delta)^{\alpha-1})$ is the polyharmonic operator. A continuation method together with a priori estimates will be exploited. Moreover, we prove uniqueness for the particular case $\alpha = 2, \beta = 1$ and $p, q > 1$.

1. Introduction

We will be concerned with the study of existence of solutions to the following Lane–Emden type system [18,22,26]

$$\begin{cases} (-\Delta)^\alpha u = |v|^{q-1} v & \text{in } \Omega \subset \mathbb{R}^N \\ (-\Delta)^\beta v = |u|^{p-1} u & \\ \dfrac{\partial^r u}{\partial \nu^r} = 0, \ r = 0, \ldots, \alpha - 1, \ \text{on } \partial \Omega \\ \dfrac{\partial^r v}{\partial \nu^r} = 0, \ r = 0, \ldots, \beta - 1, \ \text{on } \partial \Omega \end{cases} \tag{1}$$

where Ω is a smooth bounded domain, $N > \max\{2\alpha, 2\beta\}$, ν is the outward pointing normal, $\alpha, \beta \in \mathbb{N}$, $\alpha, \beta \geq 1$ and $(-\Delta)^\alpha = -\Delta((-\Delta)^{\alpha-1})$ is the polyharmonic operator. On the one hand, systems in which two

E-mail address: d.schiera@uninsubria.it.

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nonlinear PDE are coupled in a Hamiltonian fashion, namely

$$\begin{cases} L_1 u = \dfrac{\partial H}{\partial v}(u,v) \\ L_2 v = \dfrac{\partial H}{\partial u}(u,v) \end{cases}$$

with given function $H : \mathbb{R}^2 \to \mathbb{R}$ and operators L_1, L_2, have attracted a lot of attention in the last two decades both from the Mathematical as well as Physical point of view, as those models describe, among many others, nonlinear interaction between fields, see [6,36]. On the other hand, the polyharmonic operator appears in many different contexts, such as in the modeling of classical elasticity problems (in particular suspension bridges [20]), as well as Micro Electro-Mechanical Systems (MEMS), see [13] and references therein.

In order to present our results, let us first briefly survey some existing literature about Lane–Emden type systems. Consider

$$\begin{cases} -\Delta u = |v|^{q-1}v & \\ -\Delta v = |u|^{p-1}u & \Omega \subset \mathbb{R}^{\mathbb{N}} \\ u = v = 0 & \partial\Omega \end{cases} \tag{2}$$

where $N > 2$ and Ω is a smooth bounded domain, which is (1) in the particular case $\alpha = \beta = 1$. This problem turns out to be variational, namely weak solutions to (2) are critical points of the functional

$$I(u,v) = \int \nabla u \nabla v - \frac{1}{p+1}\int |u|^{p+1} - \frac{1}{q+1}\int |v|^{q+1}. \tag{3}$$

In the case $u = v$, $p = q$, (2) reduces to the single equation

$$\begin{cases} -\Delta u = |u|^{p-1}u & \Omega \subset \mathbb{R}^{\mathbb{N}} \\ u = 0 & \partial\Omega \end{cases} \tag{4}$$

and the corresponding functional is given by

$$I(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}. \tag{5}$$

The existence of weak solutions to (4) can be proved by exploiting the Mountain Pass Theorem [2] if $p \in (1, \frac{N+2}{N-2})$, whereas if $p \geq \frac{N+2}{N-2}$ then no positive solutions do exist, due to the Pohozaev identity [28]. Therefore, the value $\frac{N+2}{N-2}$ is the threshold between existence and non existence of solutions to (4). Note that $\frac{N+2}{N-2} = p^* - 1$ where p^* is the critical Sobolev exponent.

When considering the case of systems of the form (2), the situation changes deeply, since the quadratic part of the functional (3) turns out to be strongly indefinite and, as a consequence, classical variational results such as the Mountain Pass Theorem do not apply. However, it is still possible to get existence of solutions by exploiting the Linking Theorem of Benci and Rabinowitz [7] and reduction methods, see [18,22] as well as the surveys on this topic [11,30] and the references therein. More precisely, the following hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \tag{6}$$

which has been introduced by Mitidieri [26], plays the role of critical threshold for (2), namely we have existence of solutions below and non existence above (6). Note that the energy functional (3) is well defined on $W^{1,s} \times W^{1,t}$ with $\frac{1}{s} + \frac{1}{t} = 1$ provided that $\frac{1}{p+1} \geq \frac{1}{s} - \frac{1}{N}$ and $\frac{1}{q+1} \geq \frac{1}{t} - \frac{1}{N}$, due to the Sobolev embeddings [1]. Therefore,

$$\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{N} = \frac{N-2}{N}$$