Nonlinear Analysis 168 (2018) 154–175

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Global solution for a non-local eikonal equation modelling dislocation dynamics

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ARTICLE INFO

Article history:
Received 8 May 2017
Accepted 30 November 2017
Communicated by Enzo Mitidieri

MSC:
35A01
74G25
35F20
35F21
70H20
35Q74

Keywords:
Hamilton–Jacobi equation
Non-local eikonal equation
Non-local transport equation
Viscosity solution
Dislocation dynamics

ABSTRACT

In this paper, we study a non-local eikonal equation that arises from the theory of dislocation dynamics. For this equation, the global existence and uniqueness are available only for Lipschitz continuous viscosity solutions in some particular cases. Based on a new gradient entropy estimate, we prove the global existence of a continuous viscosity solution of this equation, considering non-signed velocity.

1. Introduction and main results

1.1. Presentation and physical motivations

A perfect crystal, for small deformations, is well described by the equations of linear elasticity. The real crystals contain in particular some line defects called dislocations (we refer to Hirth et al. [19] for a physical presentation of dislocations). Dislocation dynamics is the main explanation of metallic plastic deformation. When we apply an exterior stress, these dislocation lines can move in a slip plane of the crystal. The dislocation dynamics is given by a normal velocity which is proportional to the Peach–Koehler force and that can be calculated from the equations of linear elasticity. Here, we are interested in the study of a one-dimensional sub-model of the particular model proposed initially in the two-dimensional case and introduced by Rodney, Le Bouar, Finel [22]. More precisely, we consider a simple geometry where dislocations

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https://doi.org/10.1016/j.na.2017.11.012
0362-546X/

are parallel lines moving in the same plane (xy). Which plane is embedded in a three-dimensional elastic crystal. The particular geometry of this problem leads to study a one-dimensional model given by the following non-local eikonal equation

$$\begin{cases} \partial_t u(x,t) = (c_0(\cdot) \star u(\cdot,t)(x))|\partial_x u(x,t)| & \text{in } \mathbb{R}\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \tag{1.1}$$

where $T > 0$, the solution u is a real-valued function, $\partial_t u$ and $\partial_x u$ stand, respectively, for its time and space derivatives. Here, $\star$ denotes the convolution in space and $c_0(\cdot)$ is a kernel which depends only on the physical properties of the crystal and the choice of the dislocation whose evolution follows. Note that, in the special case of the application to dislocations, the kernel c_0 does not depend on time and it can change sign. Furthermore, in this non-local and non-monotone "geometrical" equation the velocity can change sign. Therefore, the principle inclusion, which plays a central role in the "level-set approach", does not hold and then the uniqueness of the solution cannot be proved via standard viscosity solutions methods. We refer the reader to [5,14,15] for a complete overview of viscosity solutions.

Although this model of dislocation dynamics seems very simple, there are only few existence and uniqueness known results. We point out that, in the case of nonnegative velocities, the global existence and uniqueness were first obtained by Alvarez et al. [2] and then by Barles et al. [9] using different arguments. These uniqueness results were recently extended by Barles et al. in [8], using a new approach that allows to lessen the assumptions of [2,9]. Moreover, the proof proposed in [8] is simpler than that in [2,9] and requires a mild regularity on the velocity.

In the general case, with unsigned velocity and in higher dimensional space, this problem has first been investigated by Alvarez et al. in [3] where a short time existence and uniqueness result was proved under the assumption that the initial position of the dislocation is a Lipschitz graph. Then, using the stability result of Barles [6] in the framework of L^1-viscosity solutions, global existence results of weak discontinuous viscosity solutions were obtained by Barles et al. in [7]. Moreover, two cases in which uniqueness can be obtained were presented in [7] that contains, as well, an interesting counter-example on the uniqueness of weak solutions. We also refer to Barles [4] for another counter-example on the uniqueness of discontinuous viscosity solution. Let us mention that, the assumptions used in [7] were relaxed in [8]. Recently, by considering the one dimensional equation and based on a new BV estimate, global existence results for some weak discontinuous viscosity solutions were obtained in [10,11].

From the numerical point of view, a new fast-marching algorithm for the eikonal equation in the case where the velocity can change sign, was proposed by Forcadel in [17], where the author proved the convergence and the comparison principle of the algorithm. Interesting applications of this algorithm, for dislocation dynamics computation and image segmentation, were also shown in [13] and [18] respectively.

1.2. Main results

In this paper, we present a global existence result for the eikonal equation (1.1) with a velocity that can change sign and considering a class of continuous initial data. More precisely, we consider the case when the kernel satisfies the following assumptions

$$c_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{1.2}$$

and

$$\mathcal{F}(c_0)(\xi) \le 0, \quad \forall \xi \in \mathbb{R}, \tag{1.3}$$

where $\mathcal{F}$ stands the Fourier transform. We note that, physically, this condition on the kernel is satisfied in the case of isotropic elasticity for one of Peierls–Nabarro model, see [3, Proposition 6.1], where an explicit