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A finite dimensional approach to light rays in General Relativity

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ABSTRACT

We propose a finite dimensional setup for the study of lightlike geodesics starting orthogonally to a spacelike $(n-2)$-submanifold and arriving orthogonally to the time-slices of an $(n-1)$-dimensional timelike submanifold of a n-dimensional spacetime. Under a transversality and a nonfocality assumption, we prove a finite dimensional reduction of a general relativistic Fermat principle, and we give a formula for the Morse index. We present some applications to bifurcation theory, and we conclude the paper with the discussion of some examples that illustrate our results.

1. Introduction

In a general relativistic spacetime, light rays from an extended light source to an extended receiver (a screen) are modeled by lightlike geodesics that are orthogonal at their endpoints to two given spacelike submanifolds. Considering the worldline Γ of a receiver, and assuming that this set is a stably causal Lorentzian hypersurface of the spacetime, i.e., a hypersurface that admits a (smooth) time function as a Lorentzian manifold of its own, then the light rays starting orthogonally to the initial submanifold and terminating orthogonally to the time slices of Γ are characterized by Fermat's principle as stationary points of the arrival time functional, see Ref. [17].

This variational principle lacks regularity, in that the set of trial paths on which the arrival time is to be considered, which consists of all (future pointing, piecewise smooth) lightlike curves between the source and the receiver, does not admit a differentiable structure. This is an obstruction to the application of analytical techniques, such as Lusternik–Schnirelman theory, Morse theory, or bifurcation theory, whose setup requires a quite elaborate functional framework, and it makes unfeasible the use of singularity theory. In this paper we propose a finite dimensional (smooth) reduction of the Fermat principle, which is suited

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to give a local description of the orthogonal light rays near a degenerate one, and that in particular allows a direct application of bifurcation theory and singularity theory to study the caustics. This aims naturally at establishing multiplicity results for light rays between sources and observers, which model the so-called multiple image effect and the gravitational lensing phenomenon in General Relativity. The interested reader will find a very extensive literature on the subject, see for instance Ref. [13], or the living review [16] for a detailed account of the recent bibliography. Fermat's principle in general relativistic optics, and its applications to gravitational lensing are discussed thoroughly in the monograph [15]. Important aspects of the theory of gravitational lensing are presented in the survey [18]. Of course, a finite dimensional approach to light rays can be obtained using the normal exponential map. However, an essential point of the finite dimensional reduction which is presented here is the fact that it preserves the variational structure of the problem, and therefore is also suited for developing Morse theoretical techniques, or to assess stability results.

Our model proposes to study orthogonal light rays using the arrival time functional restricted to the finite dimensional manifold of lightlike geodesics issuing orthogonally to the initial spacelike manifold $\mathcal{P}_0$ (the extended light source), and arriving transversally onto a timelike hypersurface Γ (the worldline of an extended receiver). The arrival time of lightlike geodesics in Lorentzian geometry plays the same role that the (squared) distance function plays in the study of focal properties of submanifolds in Riemannian geometry. Generically, the focal set is the bifurcation or catastrophe set for the family of distance functions from ambient points, see [21,22]. We assume that, with the induced metric, Γ is a *stably causal* Lorentzian manifold in itself, i.e., it admits a smooth time function $T : \Gamma \to \mathbb{R}$; for $\tau \in \mathbb{R}$, Γ_τ will denote the time slice $T^{-1}(\tau)$. In this situation, the arrival time is a smooth function in the space of light rays issuing from $\mathcal{P}_0$ and arriving on Γ, and under a nonfocality assumption (Section 2.3), its critical points correspond to light rays that arrive orthogonally to the time slices of Γ (Theorem 3.1). Moreover, a second order variational principle also holds, in the following sense. First, nondegenerate critical points p of the arrival time functional correspond exactly to nondegenerate orthogonal light rays ℓ_p. Second, the Morse index of the critical point p is equal to the Morse index of the geodesic action functional at ℓ_p minus the *focal index* of ℓ_p. Such difference can be easily interpreted *geometrically*: it is the so-called *concavity index* that appears in the Morse index theorem for orthogonal geodesics (see Theorem 2.1), and it is given in terms of the second fundamental form of the target manifold, computed in a space associated to the $\mathcal{P}_0$-Jacobi fields. Our Morse index theorem provides a *physical* interpretation of the concavity index form along an orthogonal lightlike geodesic, which is now seen as the second variation of the arrival time functional.

As to the nonfocality assumption needed for our theory, a simple counterexample shows that it cannot be omitted, see Example 1. A discussion on this assumption is presented in Section 3.2, where we show that focal points correspond indeed to focusing points of families of lightlike geodesics issuing orthogonally from $\mathcal{P}_0$. We will also show here that the nonfocality assumption can be replaced by the assumption that τ, defined in (3.1), has only nondegenerate critical points, see Corollary 3.4.

Also, using the variational principle introduced in this paper, we give a notion of stability for light rays between an extended light source and an extended receiver. By the Morse index theorem, stability is equivalent to the positive-definiteness of the concavity index form (Corollary 4.8). In the case of lightlike geodesics between a pointwise source and a pointwise observer, the Morse index is always given by the number of conjugate instants along the ray, which happens to be independent on the orientation of the geodesic, i.e., a future-pointing is stable if and only if its backwards past-point reparameterization is stable. One of the interesting consequences of our theory is the fact that, when one considers extended source and receiver, the Morse index and the notion of stability *do* indeed depend on the time orientation of the lightlike geodesic. Thus, an observer may have different notions of stability for the image of an extended source on an extended receiver depending on whether he/she is located at the source or at the target. This is discussed in Section 4.3, where a formula relating the Morse indices of lightlike geodesics and its backwards reparameterization is given (Proposition 4.6). Explicit examples of situations of orthogonal lightlike geodesics