Nonlinear Analysis 167 (2018) 29–50

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Abstract Hardy spaces with variable exponents

Yin Liu, Jiman Zhao*

School of Mathematical Sciences, Beijing Normal University, Key Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

ARTICLE INFO

Article history:
Received 21 April 2017
Accepted 21 October 2017
Communicated by Enzo Mitidieri

MSC:
42B30
42B35

Keywords:
Variable exponent Hardy spaces
Atomic decomposition
Molecular decomposition
Bilinear operators

ABSTRACT

In this paper, we introduce the definition of abstract Hardy spaces with variable exponents via the atomic decomposition and molecular decomposition. We prove the continuity from our variable Hardy space $H^{p(\cdot)}_{ato}$ into $L^{p(\cdot)}$, where $p(\cdot):\mathbb{R}^n \to (0,1]$, $0 < p_- \leqslant p_+ \leqslant 1$. Moreover, we investigate the bilinear theory. Finally, we give an application of the abstract Hardy spaces with variable exponents.

1. Introduction and preliminaries

Function spaces with variable exponents have received more and more attention in recent years, and have been extensively studied in harmonic analysis, fluid dynamics, image processing, partial differential equations and variational calculus, see [1–3,13,17,18,20–22,24,28,33,37,42,45–50,53], etc., and the references therein. Variable exponent Lebesgue spaces are a generalization of the classical $L^p(\mathbb{R}^n)$ spaces, via replacing the constant exponent p with an exponent function $p(\cdot)$: $\mathbb{R}^n \to (0,\infty)$, that is, they consist of all measurable functions f such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}dx < \infty.$$

These spaces were introduced by Birnbaum–Orlicz [11] and Orlicz [40] (see also Luxemburg [36] and Nakano [38,39]) and then systematically developed in [18,20].

In 2012, Nakai and Sawano [37] extended the theory of variable Lebesgue spaces by studying the Hardy spaces with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$, in which they established the atomic characterizations of

* Corresponding author.
E-mail addresses: lylight@mail.bnu.edu.cn (Y. Liu), jzhao@bnu.edu.cn (J. Zhao).

https://doi.org/10.1016/j.na.2017.10.011

$H^{p(\cdot)}(\mathbb{R}^n)$. Later, Sawano [43] extended the atomic characterization of $H^{p(\cdot)}(\mathbb{R}^n)$ in [37], which also improved the corresponding result in [37], and provided some applications, such as the boundedness of the fractional integral operator and the commutators generated by singular integral operators and BMO functions, and an Olsen's inequality. Independently, Cruz-Uribe and Wang [19] also investigated the variable exponent Hardy space with some slightly weaker conditions than those used in [37]. In 2016, Rocha [41] studied the solution of the equation $\triangle^m F = f$ for $f \in H^{p(\cdot)}(\mathbb{R}^n)$, where $H^{p(\cdot)}(\mathbb{R}^n)$ is defined as in [37]. Recently, Yang and Zhuo [54] studied the variable Hardy spaces associated with non-negative self-adjoint operators satisfying Gaussian estimates by an atomic characterization, and they also gave its maximal function characterizations.

On the other hand, the real variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ on the Euclidean space $\mathbb{R}^n$ has attracted many attentions, which initiated by Stein and Weiss [44], and then systematically researched by Fefferman and Stein [25]. Fefferman and Stein [25] brought real variable methods into this subject, eventually, the evolution of their ideas led to the molecular or atomic characterizations and the applications of Hardy spaces, see [5,6,10,23,34,35,51,54], etc., and the references therein. Moreover, the molecular and the atomic characterizations enabled the extension of the real variable theory of Hardy spaces on $\mathbb{R}^n$ to a more general setting for function spaces, the spaces of homogeneous type [14,15,31].

About the abstract Hardy spaces, Bernicot and Zhao [8,9] studied a kind of new abstract Hardy spaces $H^1_{\epsilon,mol}(X)$ (X is a space of homogeneous type) which keep the main properties of the classical Hardy spaces H^1. In the present paper, we give a research about the abstract Hardy spaces with variable exponents. Since we have no idea about how to prove some important properties of variable exponent function space in X, the study of this project is in the n-dimensional Euclidean space $\mathbb{R}^n$. However, our atoms and molecules are more general than the classical ones.

In [9], as an application of the new abstract Hardy spaces of [8], Bernicot and Zhao obtain the maximal L^q regularity for Cauchy problem. In this paper, since $p(\cdot) : \mathbb{R}^n \to (0,1)$, as an application of the abstract Hardy spaces with variable exponents, we only obtain the $H^{p(\cdot)}_{\epsilon,mol}(J\times\mathbb{R}^n) - L^{p(\cdot)}(J\times\mathbb{R}^n)$ boundedness of the operator T, where $Tf(t,x) = \int_0^t [Le^{(t-s)L} f(s,\cdot)](x)ds$ (associated to the maximal regularity of the Cauchy problem).

Now, we recall some notation and notions on variable exponent Lebesgue spaces.

The measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ is called the variable exponent. For a measurable subset $E \subset \mathbb{R}^n$, we write

$$p_+(E) \equiv \sup_{x\in E} p(x), \quad p_-(E) \equiv \inf_{x\in E} p(x).$$

We abbreviate $p_+(\mathbb{R}^n)$ and $p_-(\mathbb{R}^n)$ to p_+ and p_-, respectively, and we always assume that

$$0 < p_- \leqslant p_+ < \infty. \tag{1.1}$$

For a measurable function f, let

$$\|f\|_{L^{p(\cdot)}} \equiv \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

If $0 < \iota \leqslant \underline{p} \equiv \min(p_-, 1)$, it is easy to see the following properties:
(i) (Positivity) $\|f\|_{L^{p(\cdot)}} \geqslant 0$, and $\|f\|_{L^{p(\cdot)}} = 0 \Leftrightarrow f \equiv 0$.
(ii) (Homogeneity) $\|cf\|_{L^{p(\cdot)}} = |c|\cdot\|f\|_{L^{p(\cdot)}}$ for $c \in \mathbb{C}$.
(iii) (The ι-triangle inequality) $\|f+g\|^{\iota}_{L^{p(\cdot)}} \leqslant \|f\|^{\iota}_{L^{p(\cdot)}} + \|g\|^{\iota}_{L^{p(\cdot)}}$.
We note that the ι-triangle inequality follows from the following identity:
For $0 < p_- < 1$, choosing ι such that $0 < \iota \leqslant p_-$, then we have

$$\|f^{\iota}\|^{\frac{1}{\iota}}_{L^{\frac{p(\cdot)}{\iota}}} = \inf\left\{\lambda^{\frac{1}{\iota}} > 0 : \int_{\mathbb{R}^n} \left(\frac{|f^{\iota}(x)|}{\lambda}\right)^{\frac{p(x)}{\iota}} dx \leqslant 1\right\} = \|f\|_{L^{p(\cdot)}}. \tag{1.2}$$