Nonlinear Analysis 167 (2018) 51–66

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Boundary blow-up solutions to the k-Hessian equation with singular weights

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ARTICLE INFO

Article history:
Received 21 June 2017
Accepted 3 November 2017
Communicated by Enzo Mitidieri

MSC:
34B18
34B15
34A34

Keywords:
k-Hessian equation
Boundary blow up
Sub-supersolution method
k-convex solution

ABSTRACT

In this paper we study the k-convex solutions to the boundary blow-up k-Hessian problem

$$S_k(D^2u) = H(x)u^p \text{ for } x \in \Omega,\ u(x) \to +\infty \text{ as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

Here $k \in \{1, 2, \dots, N\}$, $S_k(D^2u)$ is the k-Hessian operator, and Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ ($N \geq 2$). We show the existence, nonexistence, uniqueness results, global estimates and estimates near the boundary for the solutions. Our approach is largely based on the construction of suitable sub- and super-solutions.

1. Introduction

Consider the boundary blow-up solutions for the k-Hessian equation

$$S_k(D^2u) = H(x)u^p \text{ in } \Omega,\ u = +\infty \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N(N \geq 2)$, and $H(x)$ is smooth positive function. The boundary blow-up condition $u = +\infty$ on $\partial\Omega$ means

$$u(x) \to +\infty \text{ as } d(x) := \operatorname{dist}(x, \partial\Omega) \to 0.$$

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https://doi.org/10.1016/j.na.2017.11.001
0362-546X/

$S_k(D^2u)(k \in \{1, 2, \dots, N\})$ denotes the kth elementary symmetric function of the eigenvalues of D^2u, the Hessian of u, i.e.

$$S_k(D^2u) = S_k(\lambda_1, \lambda_2 \dots, \lambda_N) = \sum_{1 \le i_1 < \dots < i_k \le N} \lambda_{i_1} \dots \lambda_{i_k},$$

where $\lambda_1, \lambda_2 \dots, \lambda_N$ are the eigenvalues of D^2u (see [4,27,9]).

It is not difficult to see that $\{S_k : k \in \{1, 2, \dots, N\}\}$ is a family of operators which contains Laplace operator ($k = 1$), the Monge–Ampére operator ($k = N$), and many other well known operators (see [10,11,17,19,20,21,22,29]). Comparing with numerous results on the case $k = 1$ or $k = N$ less is known about the situation $k \in \{2, \dots, N-1\}$. In recent years the k-Hessian equations have been studied by several authors in different settings (see by instance [4,8,13,24,12,7,25,14,16,15]). It is worth to mention that in [13], Y.Huang and in [7], A. Colesanti, E. Francini and P. Salani, considered the k-Hessian equation with bounded weight and obtained the existence, the uniqueness and asymptotic estimates of solution.

For $k \in \{1, 2, \dots, N\}$, let Γ_k be the component of $\{\lambda \in R^N : S_k(\lambda) > 0\} \subset R^N$ containing the positive cone

$$\Gamma^+ = \{\lambda \in R^N : \lambda_i > 0, i = 1, 2, \dots, N\}.$$

It follows that

$$\Gamma^+ = \Gamma_N \subset \dots \subset \Gamma_{k+1} \subset \Gamma_k \subset \dots \Gamma_1.$$

Definition 1.1 (*[27,9]*)**.** Let $k \in \{1, 2, \dots, N\}$,and let Ω be an open bounded subset of R^N; a function $u \in C^2(\Omega)$ is k-convex if $(\lambda_1, \lambda_2, \dots, \lambda_N) \in \bar{\Gamma}_k$ for every $x \in \Omega$, where $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues of D^2u. Equivalently, we can say that u is k-convex if $S_i(D^2u) \ge 0$ in Ω for $i = 1, \dots, k$. If $S_i(D^2u) > 0$ in Ω for $i = 1, \dots, k$, then we say that u is strictly k-convex.

Definition 1.2 (*[27,9]*)**.** Let $\Omega \subset R^N$ be an open set with boundary of class C^2. We say that Ω is strictly convex if $S_i(\kappa_1(x), \dots, \kappa_{N-1}(x)) > 0$, for $i = 1, \dots, N-1$ and for every $x \in \partial\Omega$, where $\kappa_i(x)$, $i = 1, \dots, N-1$, are the principal curvatures of $\partial\Omega$ at x.

The definition of k-convexity is naturally related to the involved operator S_k. In fact, $S_k(D^2u)$ turns to be elliptic in the class of k-convex functions and a unique k-convex solution of the related Dirichlet problem

$$\begin{cases} S_k(D^2u) = H(x) > 0 \quad in \quad \Omega, \\ u|_{\partial\Omega} = \phi \in C(\partial\Omega) \end{cases} \tag{1.2}$$

exists when Ω is strictly $(k-1)$-convex and $H \in C^\infty(\bar{\Omega})$; moreover the $(k-1)$-convexity of the domain is necessary if φ is constant (see [4]).

At the same time, we notice that the subject of blow-up solutions has received much attention starting with the pioneering work of Bieberbach [2]. It was about the following model involving the classical Laplace operator

$$\begin{cases} \Delta u = H(x) f(u) \text{ in } \Omega, \\ u = +\infty \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

with $H(x) \equiv 1$ in Ω, $f(u) = e^u$ and $N = 2$. If $f(u) = u^p$, $H(x)$ is growing like a negative power of $d(x)$ near $\partial\Omega$, the radial case was completely discussed in [6], the general case was discussed when $p > 1$ in [5], the authors obtained existence, nonexistence, uniqueness, multiplicity and estimates for all positive solutions.