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On the continuity of solutions of the nonhomogeneous evolution $p(x,t)$-Laplace equation

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ABSTRACT

We study the continuity of solutions of the homogeneous Dirichlet problem for the equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right) = f \quad \text{in } Q_T = \Omega \times (0,T],$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain with the boundary $\partial\Omega \in C^{2+\alpha}$ and $p(x,t)$ is a given function such that $\frac{2n}{n+2} < p^- \leq p(x,t) \leq p^+ < \infty$, $p^\pm = const$. It is shown that if p, f are Hölder-continuous in $\overline{Q}_T$ and $|\nabla u_0|^{p(x,0)} \in L^1(\Omega)$, then $u \in C^{1,\frac{1}{2}}(\overline{Q}_T \cap \{t \geq \epsilon\})$ for every $\epsilon > 0$. Moreover, if f is Lipschitz-continuous in t, $p \equiv p(x)$, $\nabla p \in L^\infty(\Omega)$ and either $p(x) \geq p^- > 2$, or $\frac{2n}{n+2} < p^- \leq p(x) \leq p^+ < 2$, then the solutions are Lipschitz-continuous in time in $\overline{Q}_T \cap \{t \geq \epsilon\}$.

1. Introduction

This work addresses the regularity properties of solutions of the problem

$$\begin{aligned}
&u_t - \operatorname{div}\left(|\nabla u|^{p(z)-2}\nabla u\right) = f \quad \text{in } Q_T, \quad z = (x,t) \in Q_T,\\
&u = 0 \text{ on } \Gamma_T = \partial\Omega \times (0,T),\\
&u(x,0) = u_0(x) \text{ in } \Omega, \qquad n \geq 2,
\end{aligned} \tag{1.1}$$

with given $u_0(x)$, $p(z)$ and $f(z)$. In the recent decades, elliptic and parabolic equations with variable nonlinearity were gaining ground in the mathematical modeling of complex real world phenomena and have attracted attention of many researchers. Eq. (1.1) is a nonlinear parabolic equation which degenerates or becomes singular at the points where $\nabla u = 0$, but the variable nonlinearity prevents one from a straightforward application of the results and techniques proper to the nonlinear equations with constant p. Because of the nonlinearity one cannot expect the existence of smooth solutions, therefore the solution

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of problem (1.1) is understood in a weak sense as an element of a Sobolev space with variable exponent, while its time derivative is an element of the dual space. The theorems of existence and uniqueness of weak solutions of problem (1.1) were proved in [4] for a single equation and in [12] for systems of evolution $p(x,t)$-Laplace equations, the problem with the nonhomogeneous Dirichlet boundary condition was studied in [13].

The intrinsic regularity properties of the local weak solutions of equations of the type (1.1) have been studied by several authors. It is known that the local solutions of Eq. (1.1) with $f \equiv 0$ are locally Hölder-continuous [3,7,22], provided that the variable exponent $p(x,t)$ is continuous with the logarithmic modulus of continuity (see condition (1.5)). Local Hölder continuity of the gradients is established in [7] for the local weak solutions of Eq. (1.1) with $f \equiv 0$ and in [23] for the local solutions of the nonhomogeneous equation in divergence form

$$u_t - \operatorname{div}\left(|\nabla u|^{p(z)-2}\nabla u\right) = -\operatorname{div}\left(|\mathbf{f}|^{p(z)-2}\mathbf{f}\right). \tag{1.2}$$

The underlying idea behind the local considerations is the use of the intrinsic geometry of the nonlinear equation — see [7,8,11,21] for an insight into the method and further references.

The Hölder continuity of the gradients was first proved in [2] for the solutions of nondegenerate parabolic PDEs with $p(x,t)$-growth under the assumption that the variable exponent $p(x,t)$ is Lipschitz-continuous in x and Hölder-continuous in t. The regularity of solutions of elliptic equations and minimizers of functionals with nonstandard growth has been studied very actively. We refer the reader to [1,18] for a review of the literature on this issue and to [9] for the proof of local Hölder continuity of minimizers of the functionals with $p(x)$-growth and Hölder-continuous $p(x)$.

For equations of the type (1.1) and (1.2) with variable p regularity in time is less studied. Apart from the local Hölder continuity of solutions of the homogeneous equation (1.1) [3], it is proved in [19] that in the case $n = 1$ and $p = p(x) > 1$ the homogeneous Dirichlet problem for the equation

$$u_t = (|u_x|^{p(x)-2}u_x)_x + f(x,u,u_x) \tag{1.3}$$

admits globally Lipschitz-continuous solutions. Moreover, it is shown that in the singular case $p(x) \in (1,2)$ u_{xx} is essentially bounded. The solution of Eq. (1.3) is obtained in [19] as the limit of a sequence of classical solutions of the regularized problems. To estimate the oscillation in time, the classical maximum principle is applied to the solutions of auxiliary ultra-parabolic problems with two time variables, which allows one to avoid differentiation of the equation in t. We refer also to [20] where the same method is used to obtain Lipschitz-continuous solutions of the anisotropic parabolic equations

$$u_t = \operatorname{div}\left(\sum_{i=1}^{n} |D_i u|^{p_i-2} D_i u\right)$$

with constant exponents $p_i \in (1,\infty)$. Global higher regularity of solutions of systems of singular parabolic equations of p-Laplacian type with constant $p \in (1,2)$ is studied in [10].

The aim of the present work is to find sufficient conditions of the uniform with respect to $x \in \overline{\Omega}$ Hölder and Lipschitz-continuity in time of solutions of problem (1.1).

Prior to formulating the results, we need to introduce the spaces of functions the solutions of problem (1.1) belong to.

1.1. Lebesgue and Sobolev spaces with variable exponents

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz-continuous boundary $\partial\Omega$. Given a function $r(z) : \Omega \mapsto [p^-, p^+] \subset (1,\infty)$, we define the set

$$L^{r(\cdot)}(\Omega) = \left\{ f : \Omega \mapsto \mathbb{R} : f \text{ is measurable on } \Omega,\ \rho_{r(\cdot),\Omega}(f) = \int_{\Omega} |f|^{r(z)}\,dz < \infty \right\}.$$