Nonlinear Analysis 166 (2018) 19–47

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Existence and regularity of nonlinear advection problems

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ARTICLE INFO

Article history:
Received 3 June 2017
Accepted 7 October 2017
Communicated by Enzo Mitidieri

Keywords:
Elliptic PDEs
Dead core
Positive solutions

ABSTRACT

This paper examines the existence and regularity of classical positive solutions of $-\Delta u = |\nabla u|^p$ on a bounded domain in $\mathbb{R}^N$ with $0 < p < 1$. This appears to be the first paper to discuss dead core solutions for a PDE with a nonlinear advection term. We also give a Liouville-type result for supersolutions on an exterior domain, asymptotics of solutions as $p \nearrow 1$ and $p \searrow 0$, as well as various extensions of the PDE.

1. Introduction

In this article we examine the existence and regularity of positive solutions of the following

$$\begin{cases} -\Delta u = |\nabla u|^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $p > 0$. We also consider some variations on (1).

Our motivation for studying this problem comes from the well-studied Lane-Emden equation given by

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p > 1$. This equation (and variations of) have played a central role in the development of the calculus of variations, bifurcation theory, the role of the critical Sobolev exponent, and Pohazaev identity, see for instance [14,28,32,45,48]. In the case of p supercritical the existence versus nonexistence of solutions of (2) on non star shaped domains becomes a very nontrivial issue, see for instance [20,24,44]. For versions of (2) on exterior domains see [21–23].

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https://doi.org/10.1016/j.na.2017.10.007

Consider the variation of (2) given by $\Delta u = u^p$ in the case of $0 < p < 1$. In this case there exist dead core solutions; i.e., solutions which (in this case) are zero on a closed set with nonempty interior, see [47].

We now return to (1). The first point is that it is a non variational equation and hence various standard tools are not available anymore. Some relevant monographs for this work include [29,33,39]. We now point out a basic feature of (1). In the case of $p \geq 1$ there is no non-zero classical solution of (1) since we can re-write the equation as $-\Delta u - b(x) \cdot \nabla u(x) = 0$, where $b(x) = |\nabla u|^{p-2}\nabla u$. Hence we can apply the maximum principle, provided $b(x)$ is sufficiently regular, to see that $u = 0$. So with this in mind we restrict our attention to classical solutions of (1) in the case of $0 < p < 1$. One can also examine suitably singular solutions of (1) in the range of $\frac{N}{N-1} < p < 2$ and $p > 2$, see Example 1 and forthcoming paper [1].

We recall some background regarding (1). Many people have studied boundary blow up solutions of the variation of (2) given by $\Delta u = u^p$ (and generalizations of). Similarly people have studied boundary blow up versions of (1) where one removes the minus sign in front of the Laplacian; see for instance [40,49]. See [6–13,19,25,26,30,31,34,35,41–43,46] for more results on equations similar to (1). Needless to say, this list is incomplete. We remark that several papers consider a much more general PDE than (1). Those papers tend to discuss existence of solutions which is not an issue in (1) since the trivial solution is always a solution. This paper appears to be the first to discuss dead core solutions to PDEs with a nonlinear advection term.

We now give a brief summary of the results in papers listed above. [6] studied positive weak solutions of $-\Delta u = B|\nabla u|^2/u + f$ for positive B and non-negative f. It was shown that positive solutions exist iff $B < 1$. [7] considers non-negative weak solutions of $-\Delta u + |\nabla u|^2 u^{-\gamma} = f > 0$, where γ is a positive constant. The main result is that such solution exists iff $\gamma < 2$. [8] and [9] consider $-\Delta u = d(x)u + \mu(x)|\nabla u|^2 + h(x)$ and its generalization $-Lu = H(x, u, \nabla u) + h(x)$, where L is a second-order linear elliptic operator and H grows at most quadratically in $|\nabla u|$ and satisfies other appropriate conditions. [10] demonstrates existence of weak solutions of $Au + g(x, u, \nabla u) = h(x)$, where A is a quasi-linear elliptic operator, the nonlinearity g satisfies a growth condition in $|\nabla u|$ and a sign condition.

In the series of papers [11–13], they consider $-\Delta_q u + |\nabla u|^p = 0$ and its generalizations. Here $-\Delta_q$ is the q-Laplacian and $q - 1 < p < q$. A detailed study of the singularities of non-negative solutions is given. [12] gives a classification of isolated singularities of negative solutions of this PDE, which is a generalization of (positive solutions of) (1) for the case $1 < p < 2$. The paper [19] gives existence and a classification of singular non-negative solutions of $\Delta u = u^q|\nabla u|^m$ on $\mathbb{R}^N \setminus 0$, where $q \geq 0$ and $m \in (0, 2)$.

[25,26] derived existence and a uniform L^∞ estimate of weak solutions of $-\Delta_p = |\nabla u|^p + g - div f$ and its generalization $-div\ a(x, u, \nabla u) = H(x, u, \nabla u) + g - div f$. Here $p > 1$, a satisfies certain coercive and growth conditions and H grows no faster than $|\nabla u|^p$ and satisfies a sign condition. Both f and g must be sufficiently small.

[30,31] prove existence of weak solutions of $-\Delta u = |\nabla u|^2|u|^{-\theta} + f(x)$ and generalizations. Here $\theta \in (0, 1)$ and $f \in L^m$ may not have a constant sign. [34,35] prove a priori estimates and existence of a solution for a general class of nonlinear elliptic problems $-div\ a(x, u, \nabla u) = H(x, u, \nabla u)$, where H satisfies a growth condition. In particular, it treats the case $H = |\nabla u|^p$ for $p > 1$.

[41] is a classical paper which considers $-\Delta u + H(x, \nabla u) + \alpha u = 0$, where H satisfies a growth and sign condition and $\alpha \geq 0$. As opposed to other papers in our list, existence theory for both Dirichlet and Neumann boundary conditions is given.

[42,43] study non-negative solutions of $-\Delta u + H(x, u, \nabla u) = 0$ for some H including the special cases $|\nabla u|^p$, $u^p|\nabla u|^q$ and $u^p + |\nabla u|^q$, where p and q are suitably restricted constants. In particular, when $H = |\nabla u|^p$, assume $p \in (1, 2)$. Finally, [46] contains results for $-\Delta_p u = g(u)|\nabla u|^p + f(x)$, where $p > 1$, $f \in L^m$ and g satisfies a growth condition. It is shown that a weak solution exists, and is bounded or unbounded according to whether $m > N/p$ or $m < N/p$.

We begin by looking at an example.