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Three positive solutions to an indefinite Neumann problem: A shooting method[☆]

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ABSTRACT

We deal with the Neumann boundary value problem

$$\begin{cases} u'' + \big(\lambda a^{+}(t) - \mu a^{-}(t)\big) g(u) = 0, \\ 0 < u(t) < 1, \quad \forall\, t \in [0,T], \\ u'(0) = u'(T) = 0, \end{cases}$$

where the weight term has two positive humps separated by a negative one and $g : [0,1] \to \mathbb{R}$ is a continuous function such that $g(0) = g(1) = 0$, $g(s) > 0$ for $0 < s < 1$ and $\lim_{s\to 0^+} g(s)/s = 0$. We prove the existence of three solutions when λ and μ are positive and sufficiently large.

1. Introduction

In this paper, we are interested in multiplicity of positive solutions for an indefinite Neumann boundary value problem of the form

$$\begin{cases} u'' + a(t) g(u) = 0, \\ u'(0) = u'(T) = 0, \end{cases} \tag{1.1}$$

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where the weight term $a \in L^1(0,T)$ is sign-changing and the nonlinearity $g : [0,1] \to \mathbb{R}^+ := [0,+\infty[$ is a continuous function such that

$$g(0) = g(1) = 0, \qquad g(s) > 0 \quad \text{for } 0 < s < 1, \tag{g_*}$$

and

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0. \tag{g_0}$$

In our context, a *solution* $u(t)$ of problem (1.1) is meant in the Carathéodory's sense and is such that $0 \le u(t) \le 1$ for all $t \in [0,T]$. Moreover, we say that a solution is *positive* if $u(t) > 0$ for all $t \in [0,T]$.

Our study is motivated by the results achieved in [1–3,6–8,10,11,14]. In more detail, dealing with different boundary value problems compared to the one treated here, in the quoted papers the authors established multiplicity results of positive solutions in relation to the nodal behavior of the weight $a(t)$. For that reason, we would like to pursue further the investigation of the dynamical effects produced by the weight term associated with nonlinearities satisfying conditions (g_*) and (g_0). With this purpose, first of all we notice that the search of positive solutions under these assumptions leads to some well known facts. Indeed, it is straightforward to verify that problem (1.1) has two trivial solutions: $u \equiv 0$ and $u \equiv 1$. Furthermore, by an integration of the differential equation in (1.1) and by taking into account the Neumann boundary conditions, we obtain a necessary condition for the existence of nontrivial positive solutions to problem (1.1): the weight $a(t)$ has to be sign-changing in the interval $[0,T]$. This is the peculiar characteristic which leads to call *indefinite* the problem we are studying.

Indefinite Neumann problems with a nonlinearity $g(s)$ satisfying (g_*) are a very important issue in the field of population genetics, starting from the pioneering works [4,9,12,19]. However, as far as we know, in order to achieve both results of uniqueness and multiplicity, lot of attention has been given to the proprieties of the nonlinearity $g(s)$ instead of the shape of the graph of the weight $a(t)$. Contributions in this direction are achieved in [15–17,20]. In particular, dealing with a nonlinearity $g(s)$ similar to the one taken into account in the present paper, in [17] the authors stated the following multiplicity result: if $\int_0^T a(t)\,dt < 0$ and $g(s)$ satisfies (g_*) and (g_0) along with $\lim_{s\to 0^+} g(s)/s^k > 0$ for some $k > 1$, then the Neumann problem associated with $du'' + a(t)g(u) = 0$ has at least two positive solutions for $d > 0$ sufficiently small. Instead, in the present work, we consider a different dispersal parameter d and we study the effects that an indefinite weight has on the dynamics of problem (1.1). For this reason, we suppose that the function $a(t)$ is characterized by a finite sequence of positive and negative humps. This way, our first goal is to show how the dynamics could be richer than the ones in [17] (cf. Prototypical Example).

Accordingly, throughout this paper, we will assume that

$$\begin{aligned} &\exists\, \sigma, \tau \text{ with } 0 < \sigma < \tau < T \text{ such that} \\ &a^+(t) \succ 0, \quad a^-(t) \equiv 0, \quad \text{on } [0,\sigma], \\ &a^+(t) \equiv 0, \quad a^-(t) \succ 0, \quad \text{on } [\sigma,\tau], \\ &a^+(t) \succ 0, \quad a^-(t) \equiv 0, \quad \text{on } [\tau,T], \end{aligned} \tag{a_*}$$

where, following a standard notation, $a(t) \succ 0$ means that $a(t) \ge 0$ almost everywhere on a given interval with $a \not\equiv 0$ on that interval. Moreover, given two real positive parameters λ and μ, we will consider the function

$$a(t) = a_{\lambda,\mu}(t) := \lambda a^+(t) - \mu a^-(t), \tag{1.2}$$

with $a^+(t)$ and $a^-(t)$ denoting the positive and the negative part of the function $a(t)$, respectively. In our framework, the dispersal parameter is thus modulated by the coefficients λ and μ. A weight term defined as