Nonlinear Analysis 166 (2018) 102–143

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Existence results for a Cauchy–Dirichlet parabolic problem with a repulsive gradient term

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ARTICLE INFO

Article history:
Received 8 May 2017
Accepted 26 September 2017
Communicated by Enzo Mitidieri

MSC:
35K55
35K61
35R05

Keywords:
Nonlinear parabolic equations
Unbounded data
Repulsive gradient

ABSTRACT

We study the existence of solutions of a nonlinear parabolic problem of Cauchy–Dirichlet type having a lower order term which depends on the gradient. The model we have in mind is the following:

$$\begin{cases} u_t - \text{div}\,(A(t,x)\nabla u|\nabla u|^{p-2}) = \gamma|\nabla u|^q + f(t,x) & \text{in } Q_T, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $Q_T = (0,T)\times\Omega$, Ω is a bounded domain of R^N, $N \geq 2$, $1 < p < N$, the matrix $A(t,x)$ is coercive and with measurable bounded coefficients, the r.h.s. growth rate satisfies the superlinearity condition

$$\max\left\{\frac{p}{2}, \frac{p(N+1)-N}{N+2}\right\} < q < p$$

and the initial datum u_0 is an unbounded function belonging to a suitable Lebesgue space $L^\sigma(\Omega)$. We point out that, once we have fixed q, there exists a link between this growth rate and exponent $\sigma = \sigma(q,N,p)$ which allows one to have (or not) an existence result. Moreover, the value of q deeply influences the notion of solution we can ask for.

The sublinear growth case with

$$0 < q \leq \frac{p}{2}$$

is dealt at the end of the paper for what concerns small value of p, namely $1 < p < 2$.

1. Introduction

Let $Q_T = (0,T)\times\Omega$, where Ω is a bounded domain of R^N, $N \geq 2$.

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https://doi.org/10.1016/j.na.2017.09.012
0362-546X/

We are interested in the study of existence results concerning the following nonlinear parabolic problem of Cauchy–Dirichlet type:

$$\begin{cases} u_t - \text{div } a(t,x,u,\nabla u) = H(t,x,\nabla u) & \text{in } Q_T, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the initial datum $u_0 = u_0(x)$ is a possibly *unbounded* function belonging to a suitable Lebesgue space $L^\sigma(\Omega)$, the operator $-\text{div } a(t,x,u,\nabla u)$ satisfies conditions of Leray–Lions type in the space $L^p(0,T;W_0^{1,p}(\Omega))$ with $1 < p < N$, the r.h.s. $H(t,x,\nabla u)$ is supposed to grow at most as *a power of the gradient* plus a forcing term, namely $|H(t,x,\nabla u)| \le \gamma|\nabla u|^q + f$, $\gamma > 0$, provided that $f = f(t,x)$ belongs to a suitable space $L^r(0,T;L^m(\Omega))$ and the gradient growth rate is such that $q < p$.

The model equation one has to keep in mind is the following:

$$u_t - \Delta_p u = \gamma|\nabla u|^q + f \quad \text{in } Q_T \tag{1.2}$$

where $\Delta_p v$ is the p-Laplace operator, namely $\Delta_p v = \text{div } (|\nabla v|^{p-2}\nabla v)$.

We give a very brief recall aimed at motivating both the mathematical and physical interest in the study of problem (1.2). Consider, for the sake of simplicity, the linear case $p = 2$ and thus the equation we take into account is

$$u_t - \Delta u = |\nabla u|^q + f(t,x) \quad \text{in } Q_T. \tag{1.3}$$

Eq. (1.3) can be seen, up to scaling, as the approximation in the viscous sense ($\varepsilon \to 0^+$) of Hamilton–Jacobi equations. We refer to [28] for a deeper analysis in this sense. Moreover, (1.3) is studied in the physical theory of growth and roughening of surfaces as well and it is known under the name of Kardar–Parisi–Zhang equation (see, for instance, [26,27]). We refer to [36] for a detailed overview on the several applications of (1.3). Finally, Eq. (1.2) is the simplest model for a quasilinear second order parabolic problem with nonlinear reaction terms of first order.

Here we list some previous papers and results to explain what is known in the literature.

The case $p = 2$ with *Laplace operator*, $f = 0$ and unbounded initial data belonging to Lebesgue spaces has been extensively studied in [4]. The authors provide a detailed investigation of the Cauchy problem

$$\begin{cases} u_t - \Delta u = \gamma|\nabla u|^q & \text{in } (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{1.4}$$

assuming that $q > 1$ and $\gamma \in \mathbb{R}$, $\gamma \neq 0$. Their approach to the study of (1.4) goes through semigroup theory and heat kernel estimates and points out that one is allowed to have (or not) existence of a solution u only if the gradient growth q and the integrability class of u_0 satisfy a precise relation. To be clear, they show that, for fixed value of $2 - \frac{N}{N+1} < q < 2$, u_0 has to be taken in the Lebesgue space $L^\sigma(\Omega)$ for $\sigma = \frac{N(q-1)}{2-q}$ while, if $q < 2 - \frac{N}{N+1}$, data measures are allowed. Nonexistence and nonuniqueness results are also proved for positive data $u_0 \ge 0$ whereas $\gamma > 0$, $q < 2$ and $u_0 \in L^\sigma(\Omega)$ for $\sigma < \frac{N(q-1)}{2-q}$. In addition, the authors take into account initial data in Sobolev's spaces, as well as the cases of attractive gradient ($\gamma < 0$) with positive initial data and of supernatural growth $q \ge 2$ with $\sigma \ge 1$ (in which existence fails).
Even if this reference is concerned with the Cauchy problem, several arguments are actually local in space.

We point out that the need of considering suitable data is a common feature among superlinear problems and it is not related with the nature of the superlinear term one deals with (see, for instance, [16] where the superlinear term is a power of the solution).

In a similar spirit, we refer to [5] for the study of the Cauchy–Dirichlet problem in the case of *Laplace operator*, $f = 0$ and $q > 0$, namely

$$\begin{cases} u_t - \Delta u = \gamma|\nabla u|^q & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$