Nonlinear Analysis 166 (2018) 154–180

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Eigenvalues and bifurcation for problems with positively homogeneous operators and reaction–diffusion systems with unilateral terms

Milan Kučera[a,b], Josef Navrátil[c,*]

[a] Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague, Czech Republic
[b] Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia in Pilsen, Univerzitní 8, 30614 Plzeň, Czech Republic
[c] Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Brehova 7, Prague, Czech Republic

A R T I C L E I N F O

Article history:
Received 22 August 2017
Accepted 6 October 2017
Communicated by S. Carl

Keywords:
Positively homogeneous operators
Maximal eigenvalue
Variational characterization
Global bifurcation
Reaction–diffusion systems
Unilateral sources

A B S T R A C T

Reaction–diffusion systems satisfying assumptions guaranteeing Turing's instability and supplemented by unilateral terms of type v^- and v^+ are studied. Existence of critical points and sometimes also bifurcation of stationary spatially non-homogeneous solutions are proved for rates of diffusions for which it is excluded without any unilateral term. The main tool is a general result giving a variational characterization of the largest eigenvalue for positively homogeneous operators in a Hilbert space satisfying a condition related to potentiality, and existence of bifurcation for equations with such operators. The originally non-variational (non-symmetric) system is reduced to a single equation with a positively homogeneous potential operator and the abstract results mentioned are used.

1. Introduction

The original goal of this paper was a study of an influence of unilateral terms of type v^-, v^+ to bifurcation of stationary spatially non-homogeneous solutions of reaction–diffusion systems exhibiting Turing's diffusion driven instability. The systems discussed have the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \triangle u + b_{11} u + b_{12} v + n_1(u,v) \\ \frac{\partial v}{\partial t} &= d_2 \triangle v + b_{21} u + b_{22} v + n_2(u,v) + \hat{g}_-(x, v^-) - \hat{g}_+(x, v^+) \end{aligned} \quad \text{in } \Omega \times [0,\infty), \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^m$ with a Lipschitz boundary, d_1, d_2 are positive parameters, b_{ij} are real constants, $n_1, n_2 : \mathbb{R}^2 \to \mathbb{R}$ are small nonlinear perturbations, v^+, v^- denote the positive and the negative

* Corresponding author.
E-mail addresses: kucera@math.cas.cz (M. Kučera), Josef.Navratil@fjfi.cvut.cz (J. Navrátil).

https://doi.org/10.1016/j.na.2017.10.004
0362-546X/

part of v, respectively, and $\hat{g}_-, \hat{g}_+ : \Omega \times [0,\infty) \to \mathbb{R}$ are functions describing certain unilateral sources and sinks, see below for more details. However, for our approach we needed a variational characterization of the largest eigenvalue of a compact positively homogeneous operator and existence of bifurcation for equations of type

$$\lambda u - Su + B(u) - N(u) = 0, \tag{2}$$

where S is a linear compact symmetric operator in a Hilbert space, B is a compact positively homogeneous operator and N is a small compact nonlinear perturbation. These results perhaps can be of a separate interest and therefore they are given in an abstract form in separate self-contained Section 4. For a variational characterization of the largest eigenvalue of a compact positively homogeneous operator B we need a certain additional assumption, namely the condition (51), which is related to potentiality. For the proof of existence of bifurcation for the equation mentioned above we need an odd multiplicity of the largest eigenvalue of S and B is supposed to be small.

Reaction–diffusion system (1) will be always supplemented by mixed boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial \vec{\nu}} = \frac{\partial v}{\partial \vec{\nu}} = 0 \quad \text{on } \Gamma_N \\ &u = v = 0 \quad \text{on } \Gamma_D, \end{aligned} \tag{3}$$

where $\vec{\nu}$ is the outer unit normal to the boundary $\partial\Omega$, $\Gamma_N, \Gamma_D \subset \partial\Omega$ are disjoint subsets of $\partial\Omega$ satisfying

$$\text{meas}_{m-1}\, \Gamma_D > 0, \quad \text{meas}_{m-1}(\partial\Omega \setminus (\Gamma_D \cup \Gamma_N)) = 0 \tag{4}$$

(the $(m-1)$-dimensional Lebesgue measure). In fact, the original model should describe a biochemical reaction of two morphogens having a positive constant equilibrium $\bar{u}$, $\bar{v}$. Shifting this positive steady state to zero, we can write the equations in the form (1), where u and v denote deviations of concentrations of the morphogens from the values $\bar{u}$, $\bar{v}$, not concentrations themselves. We will always suppose that n_j are continuously differentiable and

$$n_j(0,0) = \frac{\partial n_j}{\partial u}(0,0) = \frac{\partial n_j}{\partial v}(0,0) = 0, \; j = 1,2, \tag{5}$$

$$\begin{aligned} \det B := b_{11}b_{22} - b_{12}b_{21} > 0, \quad b_{11} + b_{22} < 0, \\ b_{11} > 0, \quad b_{22} < 0, \quad b_{12}b_{21} < 0. \end{aligned} \tag{6}$$

It is known that under the assumptions (5), (6), in the case $g_\pm = 0$ the trivial solution of the corresponding system without diffusion, i.e. ODEs obtained from (1) for $d_1 = d_2 = 0$, is asymptotically stable, but the trivial solution of the full system (1), (3) is unstable for d_1, d_2 from a certain open subset D_U of the positive quadrant $\mathbb{R}^2_+$ (Turing instability), and stable only for $(d_1, d_2) \in D_S = \mathbb{R}^2_+ \setminus \overline{D}_U$. See e.g. [13,14,4]. Our goal is to prove that for the problem with non-trivial $g_\pm$, there exist global bifurcations of spatially non-homogeneous stationary solutions in the domain D_S, where this is impossible in the case $g_\pm = 0$.

The unilateral terms $g_-(x, v^-)$ and $g_+(x, v^+)$ can model a unilateral source and sink, which is active only in points x and times t where $v(t,x) < 0$ and $v(t,x) > 0$, that means where the concentration of the second morphogen is less and larger, respectively, than $\bar{v}$. We will assume in the whole paper that $\hat{g}_-, \hat{g}_+ : \Omega \times [0,\infty) \to \mathbb{R}$ are functions satisfying Carathéodory conditions, having a derivative with respect to the second variable at zero for a.a. x $\in \Omega$ and

$$\hat{g}_\pm(x,0) \equiv 0, \quad \left.\frac{\partial \hat{g}_\pm(x,\xi)}{\partial \xi}\right|_{\xi=0} = s_\pm(x) \quad \text{for a.a. } x \in \Omega, \tag{7}$$