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Solutions of supercritical semilinear non-homogeneous elliptic problems☆

Maryam Basiri, Abbas Moameni*

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada

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ABSTRACT

Considering a semilinear elliptic equation

$$\begin{cases} -\Delta u + \lambda u = \mu g(x,u) + b(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary, we apply a new variational principle introduced in Momeni (2011, 2017) to show the existence of a strong solution, where g can have critical growth. To be more accurate, assuming $G(x,\cdot)$ is the primitive of $g(x,\cdot)$ and $G^*(x,\cdot)$ is the Fenchel dual of $G(x,\cdot)$, we shall find a minimum of the functional $I[\cdot]$ defined by

$$I[u] = \int_\Omega \mu G^*(x, \frac{-\Delta u + \lambda u - b(x)}{\mu})\, dx - \int_\Omega \mu G(x,u) + b(x) u\, dx,$$

over a convex set K, consisting of bounded functions in an appropriate Sobolev space. The symmetric nature of the functional $I[\cdot]$, provided by existence of a function G and its Fenchel dual G^*, alleviate the difficulty and shorten the process of showing the existence of solutions for problems with supercritical nonlinearity. It also makes it an ideal choice among the other energy functionals including Euler–Lagrange functional.

1. Introduction

In this paper, we are interested in showing the existence of a solution for the boundary value problem of the form

$$\begin{cases} -\Delta u + \lambda u = f(x,u,\mu) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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* Corresponding author.
E-mail addresses: maryambasiri@cmail.carleton.ca (M. Basiri), momeni@math.carleton.ca (A. Moameni).

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where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, μ and λ are two parameters, and $f : \Omega\times\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ is a Carátheodory function. There have been numerous studies about elliptic equations such as (1), and to prove the existence of solutions for this problem either topological methods or variational tools have been used in the literature (see [16,18,20,21] and the references therein). A particular case of (1), which has been investigated extensively, is the following problem

$$\begin{cases} -\Delta u + \lambda u = \mu|u|^{p-2}u + b(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\mu > 0$ and $b \in L^2(\Omega)$. Working on (2) with $\lambda = 0$ and $\mu = 1$, in [2] the author has proved that for $2 < p < 2^*$ where $2^* = 2n/(n-2)$ for $n \geq 3$ and $2^* = \infty$ for $n = 2$, problem (2) admits infinitely many weak solutions for an open dense set of b in $W^{-1,2}(\Omega)$. In [6] the author used a new version of Krasnoselskii's fixed point theorem for the sum of two operators given in [5] and proved the existence of a strong solution for (2) with λ close to zero for any $p > 2$ if $n = 3$, for $2 < p \leq (2n-4)/(n-4)$ if $n > 4$, and finally for $2 < p \leq 3$ if $n = 4$. In [3] and [17], with different methods, they also proved if $b \in L^2(\Omega)$ then problem (2) possesses infinite number of distinct solutions for $2 < p < p_N$ where p_N is a constant less than the critical exponent 2^*. In addition, in [8] and [9], they consider the problem (2) in $\mathbb{R}^n$.

Here, as a consequence of our result, we prove that if $b \in L^2(\Omega)$, $\lambda \geq 0$, and $\mu > 0$ is small then problem (2) has a strong solution for each $p > 2$ if $n \leq 4$, and for $2 < p \leq (2n-4)/(n-4)$ if $n > 4$. The importance of our result is that for this range of p, our problem includes supercritical nonlinearity as well. Furthermore, when $\partial\Omega$ is smooth enough, the new principle enables us to generalize this result by just increasing the regularity of $b(x)$. In other words, by setting $b \in W^{k,2}(\Omega)$, we can show the existence of a strong solution $u \in W^{k+2,2}(\Omega)$ for $p > 2$, if $n \leq 2(k+2)$, and for $2 < p \leq 2\big(n-2(k+1)\big)/\big(n-2(k+2)\big)$ if $2(k+2) < n \leq 2(k+2) + 4/(k-1)$. It is worth noting that the smoothness of the solution is also determined in this result.

In this paper, we also consider a more general case of (2). To unveil another aspect of the new variational principle established in [14,15] and its application, we use it as a main tool to prove that there is a solution for the following equation

$$\begin{cases} -\Delta u + \lambda u = \mu g(x,u) + b(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Note that when g has critical growth, the usual variational methods, such as mountain pass approach, do not apply here, as we do not have the compactness of Sobolev embeddings. However, as we see in this paper the new variational approach will rectify this matter. Indeed, we shall prove the following result.

Theorem 1.1. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ which $\partial\Omega$ is of class of $C^{1,1}$. Consider the problem (3), with $\lambda \geq 0$ and μ positive. Furthermore, assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carátheodory function which is increasing in u and satisfies the following growth condition:*

$$|g(x,u)| \leq a|u|^{p-1} + c, \tag{4}$$

where $a > 0$, $b \in L^d(\Omega)$ for $d \geq 2$, and

$$\begin{cases} 2 < p \leq \dfrac{(2n-2d)}{(n-2d)} & \text{if} \quad n > 2d, \\ 2 < p & \text{if} \quad n \leq 2d. \end{cases} \tag{5}$$

Then, problem (3) admits at least one solution in $W^{2,d}(\Omega)$ provided μ is small.