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Weighted vector-valued estimates for a non-standard Calderón–Zygmund operator

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In this paper, the author considers the weighted vector-valued estimates for the operator defined by

$$T_Af(x)=\mathrm{p.\,v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+1}}\big(A(x)-A(y)-\nabla A(y)\big)f(y)\mathrm{d}y,$$

and the associated maximal singular integral operator T_A^*, where Ω is homogeneous of degree zero, has vanishing moment of order one, A is a function in $\mathbb{R}^n$ such that $\nabla A\in \mathrm{BMO}(\mathbb{R}^n)$. By pointwise estimates for $\|\{T_Af_k(x)\}\|_{l^q}$ and $\|\{T_A^*f_k(x)\}\|_{l^q}$, the author obtains some quantitative weighted vector-valued bounds for T_A and T_A^*.

1. Introduction

In the remarkable work [21], Muckenhoupt characterized the class of weights w such that the Hardy–Littlewood maximal operator M satisfies the weighted L^p ($p\in[1,\infty)$) estimate

$$\|Mf\|_{L^{p,\infty}(\mathbb{R}^n,\,w)}\lesssim\|f\|_{L^p(\mathbb{R}^n,\,w)}. \tag{1.1}$$

The inequality (1.1) holds if and only if w satisfies the $A_p(\mathbb{R}^n)$ condition, that is, for $p\in(1,\infty)$,

$$[w]_{A_p}:=\sup_Q\Big(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)\mathrm{d}x\Big)^{p-1}<\infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$; and

$$[w]_{A_1}:=\sup_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)}.$$

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$[w]_{A_p}$ is called the A_p constant of w. Also, Muckenhoupt proved that M is bounded on $L^p(\mathbb{R}^n, w)$ if and only if w satisfies the $A_p(\mathbb{R}^n)$ condition. Since then, considerable attention has been paid to the theory of $A_p(\mathbb{R}^n)$ and the weighted norm inequalities with $A_p(\mathbb{R}^n)$ weights for main operators in Harmonic Analysis, see [10, Chapter 9] and related references therein.

However, the classical results on the weighted norm inequalities with $A_p(\mathbb{R}^n)$ weights did not reflect the quantitative dependence of the $L^p(\mathbb{R}^n, w)$ operator norm in terms of the relevant constant involving the weights. The question of the sharp dependence of the weighted estimates in terms of the $A_p(\mathbb{R}^n)$ constant specifically raised by Buckley [2], who proved that if $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, then

$$\|Mf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n,\, w)}. \tag{1.2}$$

Moreover, the estimate (1.2) is sharp in the sense that the exponent $1/(p-1)$ cannot be replaced by a smaller one. Hytönen and Pérez [16] showed that

$$\|Mf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} \big([w]_{A_p}[w^{-\frac{1}{p-1}}]_{A_\infty}\big)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n,\, w)} \tag{1.3}$$

where and in the following, for a weight u, $[u]_{A_\infty}$ is defined by

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)\mathrm{d}x.$$

It is well known that for $w \in A_p(\mathbb{R}^n)$, $[w^{-\frac{1}{p-1}}]_{A_\infty} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$. Thus, (1.3) is more subtle than (1.2).

The sharp dependence of the weighted estimates of singular integral operators in terms of the $A_p(\mathbb{R}^n)$ constant was much more complicated. Petermichl [23,24] solved this question for Hilbert transform and Riesz transform. Hytönen [14] proved that for a Calderón–Zygmund operator T and $w \in A_2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2(\mathbb{R}^n,\, w)} \lesssim_n [w]_{A_2} \|f\|_{L^2(\mathbb{R}^n,\, w)}. \tag{1.4}$$

This solved the so-called A_2 conjecture. Combining the estimate (1.4) and the extrapolation theorem in [8], we know that for a Calderón–Zygmund operator T, $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|Tf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} [w]_{A_p}^{\max\{1,\, \frac{1}{p-1}\}} \|f\|_{L^p(\mathbb{R}^n,\, w)}. \tag{1.5}$$

In [18], Lerner gave a simple proof of (1.4) by dominating the Calderón–Zygmund operator pointwisely using sparse operators.

Now let us consider a class of non-standard Calderón–Zygmund operators. For $x \in \mathbb{R}^n$, we denote by x_j $(1 \le j \le n)$ the jth variable of x and $x' = x/|x|$. Let Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1} and satisfy the vanishing condition that for all $1 \le j \le n$,

$$\int_{S^{n-1}} \Omega(x')x'_j \mathrm{d}x = 0. \tag{1.6}$$

Let A be a function on $\mathbb{R}^n$ whose derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$. Define the operator T_A by

$$T_Af(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} \big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y)\mathrm{d}y. \tag{1.7}$$

The maximal singular integral operator associated with T_A is defined by

$$T_A^*f(x) = \sup_{\epsilon>0} \big|T_{A,\,\epsilon}f(x)\big|,$$