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Metric gradient flows with state dependent functionals: The Nash-MFG equilibrium flows and their numerical schemes

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ABSTRACT

We investigate the convergence of a relaxed version of the *best reply* numerical schemes (also known as *best response* or *fictitious play*) used to find Nash-mean field games equilibriums. This leads us to consider evolution equations in metric spaces similar to gradient flows except that the functional to be differentiated depends on the current point; these are called equilibrium flows. We give two definitions of solutions and prove, through the introduction of a specific index Υ depending on the trajectory, that, as the time step tends to zero, the interpolated (*à la* de Giorgi) numerical curves converge to equilibrium flows. As a by-product we obtain a sufficient condition for the uniqueness of a mean field games equilibrium. We close with applications to congestion and vaccination mean field games.

1. Introduction

Let $\mathcal{X}$ be a Polish geodesic metric space (see [11] for an introduction to metric spaces) and $\mathcal{C}(\cdot,\cdot) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a functional. We investigate in this work the equation:

$$\partial_t x_t + \nabla_1 \mathcal{C}(x_t, x_t) = 0,\ x_0 = \bar{x}. \tag{1}$$

Such an equation is called an *equilibrium flow* or *partial flow* for reasons that will be made clear in the sequel.

Discrete (numerically computable) versions of this evolution equation are the numerical schemes defined by the recurrence:

$$x_0^\tau = \bar{x},\ x_{k+1}^\tau \in \underset{y\in\mathcal{X}}{argmin} \frac{d(y, x_k^\tau)^2}{2\tau} + \mathcal{C}(y, x_k^\tau), \quad k \geq 0. \tag{2}$$

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These numerical schemes are relaxed versions of the *best reply/best response/fictitious play* algorithms (see [8]); the original schemes take sometimes $\tau = \infty$ i.e., omit the first term in (2).

Our goal is to give a rigorous definition of the concept of solution of (1) and show that the numerical schemes (2) converge, when $\tau \to 0$, to a solution of (1). Finally we give examples that show that the equilibrium flows can be successfully used in the study of mean field games equilibriums.

The present work provides thus a rigorous treatment of a two-argument (partial) gradient flow distinct from previous, time-dependent approaches; this allows to obtain the convergence of the "best reply" numerical schemes (2) but also novel uniqueness results for MFGs.

These results are not available with previous techniques from [1,17,29,42,37,38], see Remark 5 after the proof of Theorem 1; in order to succeed, we introduce a new index Υ dependent of the trajectory (see definition (16)) and formalize its expected properties in assumptions $(\mathbf{A}_7)$ and $(\mathbf{A}_8)$, which we prove to be compatible with many different MFG applications (see Sections 3.1 to 3.4 and their application-dependent metric spaces that fulfill assumptions $(\mathbf{A}_7)$ and $(\mathbf{A}_8)$); the manipulation of the index Υ requires to obtain some upper bounds (see the proof of Theorem 1, one of our main results); finally we are able to obtain estimates of the partial flow divergence by making use of the hypothesis $(\mathbf{A}_8)$.

1.1. Motivation and literature review

The equilibrium in non-cooperative multi-player games are often formulated as mixed strategy Nash equilibriums (see [39]). The computation of such equilibriums and the procedure for players to reach them has been the object of many contributions and give rise to several proposals e.g., replicator dynamics and fictitious play/best reply/best response dynamics, see [19] for details.

The relatively recent introduction of the mean field games (abbreviated from now on as MFG) by Lasry and Lions [34,33,35] and conjointly by Huang, Malhamé and Caines [26,25] (see also [36,12,6,24,14,13,23,22, 41,40,15] for entry points to the literature) allow to extend this concept to games with an infinite number of players. In this context the players are considered similar (or decomposed in several classes, each with an infinite number of individuals) and an equilibrium is attained when all agents in a class use same mixed strategy, which is optimal in the Nash sense. Mixed strategies are probability measures over the state of pure strategies and thus form a metric space (we will come back later to the topological description of that space); in order to gain in generality we will suppose from now on that the space of all mixed strategies is metrizable and will be denoted $\mathcal{X}$. The cost of the individual strategy x depends on the choice of everybody else's strategy $y \in \mathcal{X}$ and is encoded through the cost function $\mathcal{C}(x, y)$. A MFG equilibrium is thus a point $x \in \mathcal{X}$ such that

$$\mathcal{C}(x, x) \leq \mathcal{C}(z, x), \forall z \in \mathcal{X}. \tag{3}$$

In this context, the relaxed best reply algorithm, which corresponds to (2) has been proposed and tested (see e.g., [8,7]) with successful results. However only very few works concern the behavior of solutions for $\tau \to 0$ in the general framework of metric spaces or the meaning to be given to the limit equation (1).

Note that when $\mathcal{C}$ is independent of the second argument, i.e.,

$$\mathcal{C}(x, y) = E(x), \tag{4}$$

the relation (2) becomes the celebrated implicit Euler-type scheme of Jordan, Kinderlehrer and Otto [28] for the definition of gradient flows in metric spaces

$$\partial_t y_t + \nabla E(y_t) = 0, \ y_0 = \bar{y}, \tag{5}$$

and received considerable attention (see [44,43,2] for instance). However, the situation when E has dependence on other variables has not been treated to the same extent and the related contributions involve