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Semilinear fractional elliptic equations with measures in unbounded domain

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ABSTRACT

In this paper, we study the existence of nonnegative weak solutions to $(E)(-\Delta)^\alpha u+h(u)=\nu$ in a general regular domain Ω, which vanish in $\mathbb{R}^N\setminus\Omega$, where $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha\in(0,1)$, ν is a nonnegative Radon measure and $h:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous nondecreasing function satisfying a subcritical integrability condition.

Furthermore, we analyze properties of weak solution u_k to (E) with $\Omega=\mathbb{R}^N$, $\nu=k\delta_0$ and $h(s)=s^p$, where $k>0$, $p\in(0,\frac{N}{N-2\alpha})$ and δ_0 denotes Dirac mass at the origin. Finally, we show for $p\in(0,1+\frac{2\alpha}{N}]$ that $u_k\to\infty$ in $\mathbb{R}^N$ as $k\to\infty$, and for $p\in(1+\frac{2\alpha}{N},\frac{N}{N-2\alpha})$ that $\lim_{k\to\infty}u_k(x)=c\mid x\mid^{-\frac{2\alpha}{p-1}}$ with $c>0$, which is a classical solution of $(-\Delta)^\alpha u+u^p=0$ in $\mathbb{R}^N\setminus\{0\}$.

1. Introduction

Let Ω be a regular domain (not necessary bounded) of $\mathbb{R}^N$ $(N\ge 2)$, $\alpha\in(0,1)$ and $d\omega(x)=\frac{dx}{1+|x|^{N+2\alpha}}$. Denote by $\mathfrak{M}^b(\Omega)$ the space of the Radon measures ν in Ω such that $\|\nu\|_{\mathfrak{M}^b(\Omega)}:=|\nu|(\Omega)<+\infty$ and by $\mathfrak{M}^b_+(\Omega)$ the nonnegative cone. The purpose of this paper is to study the existence of nonnegative weak solutions to semilinear fractional elliptic equations

$$\begin{aligned}(-\Delta)^\alpha u+h(u)&=\nu\quad\text{in }\Omega,\\ u&=0\quad\text{in }\mathbb{R}^N\setminus\Omega,\end{aligned}\tag{1.1}$$

where $\nu\in\mathfrak{M}^b_+(\Omega)$, $h:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous nondecreasing function and $(-\Delta)^\alpha$ denotes the fractional Laplacian of exponent α defined by

$$(-\Delta)^\alpha u(x)=\lim_{\epsilon\to0^+}(-\Delta)^\alpha_\epsilon u(x),$$

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where for $\epsilon > 0$,

$$(-\Delta)^{\alpha}_{\epsilon} u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|)dz$$

and

$$\chi_{\epsilon}(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

In the pioneering work [3] (also see [1]), Brezis studied the existence of weak solutions to second order elliptic problem

$$\begin{aligned} -\Delta u + h(u) &= \nu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^N (N \geq 3)$, ν is a bounded Radon measure in Ω, and the function $h : \mathbb{R} \to \mathbb{R}$ is nondecreasing, positive on $(0, +\infty)$ and satisfies the subcritical assumption:

$$\int_1^{+\infty} (h(s) - h(-s))s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

In particular case that $0 \in \Omega$, $h(s) = s^q$ and $\nu = k\delta_0$ with $k > 0$, Brezis et al. showed that (1.2) admits a unique weak solution v_k for $1 < q < N/(N - 2)$, while no solution exists if $q \geq N/(N - 2)$. Later on, Véron in [20,21] proved that if $1 < q < N/(N - 2)$, the limit of v_k is a strong singular solution of

$$\begin{aligned} -\Delta u + u^q &= 0 \quad \text{in } \Omega \setminus \{0\}, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

After that, Brezis–Véron in [4] found that the problem (1.3) admits only the zero solution if $q \geq N/(N - 2)$.

During the last years, there has also been a renewed and increasing interest in the study of linear and non-linear integro-differential operators, especially, the fractional Laplacian, motivated by various applications in physics and by important links on the theory of Lévy processes, refer to [5,7,6,8–13,15,18,19]. In a recent work, Karisen–Petitta–Ulusoy in [16] used the duality approach to study the fractional elliptic equation

$$(-\Delta)^{\alpha} v = \mu \quad \text{in } \mathbb{R}^N,$$

where μ is a Radon measure with compact support. More recently, Chen–Véron in [13] studied the semilinear fractional elliptic problem (1.1) when Ω is an open bounded regular set in $\mathbb{R}^N$ and ν is a Radon measure such that $\int_{\Omega} d^{\beta} d|\nu| < +\infty$ with $\beta \in [0, \alpha]$ and $d(x) = dist(x, \partial\Omega)$. The existence and uniqueness of weak solution are obtained when the function h is nondecreasing and satisfies

$$\int_1^{+\infty} (h(s) - h(-s))s^{-1-k_{\alpha,\beta}} ds < +\infty, \tag{1.4}$$

where

$$k_{\alpha,\beta} = \begin{cases} \dfrac{N}{N-2\alpha}, & \text{if } \beta \in \left[0, \dfrac{N-2\alpha}{N}\alpha\right], \\ \dfrac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in \left(\dfrac{N-2\alpha}{N}\alpha, \alpha\right]. \end{cases} \tag{1.5}$$

Motivated by these results and in view of the non-local character of the fractional Laplacian we are interested in the existence of weak solutions to problem (1.1) when Ω is a general regular domain, including $\Omega = \mathbb{R}^N$. Before stating our main results in this paper, we introduce the definition of weak solution to (1.1).