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Rearrangement minimization problems with indefinite external forces

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ABSTRACT

This paper discusses a rearrangement minimization problem related to a boundary value problem where the differential operator is the p-Laplacian and the external force is *indefinite*. By applying the rearrangement theory established by G. R. Burton, we show that the minimization problem has a unique solution in the weak closure of the admissible set in an appropriate function space. What makes this paper different from many others in the literature is that the admissible set here is allowed to be indefinite (takes on both positive and negative values). We show that in this case the optimal solution will be a weak limit point of the admissible set *and* cannot be a rearrangement element itself. Optimality conditions are derived based on the mean value of the generator. In case the generator is a three valued function, we shall show that the optimality conditions lead to free boundary value problems that are of interest independently. We also discuss a minimization problem where the admissible set is not seemingly related to rearrangement classes but we shall prove otherwise, and apply the earlier results of the current paper to draw interesting conclusions regarding the optimal solution. We have also discussed the case where the domain of interest is radial, and shown that the optimal solutions are radial as well.

1. Introduction

A rearrangement optimization problem is an optimization problem where the admissible set is a rearrangement class or a subset of it (e.g. the intersection of a rearrangement class with an affine subspace of finite co-dimension). In this paper we consider a problem of this type pertaining to the following boundary value problem:

$$\begin{cases} -\Delta_p u = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.1}$$

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where D is a smooth (C^2 is enough) bounded domain in $\mathbb{R}^N$, Δ_p is the classical p-Laplace operator, i.e. $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, with $1 < p < \infty$, and $f \in L^\infty(D)$. By the direct method, (1.1) has a unique solution $u_f \in W_0^{1,p}(D)$, which is also the unique minimizer of the functional:

$$E(v) \equiv \frac{1}{p}\int_D |\nabla v|^p dx - \int_D fv dx. \tag{1.2}$$

We are interested in the following minimization problem:

$$\inf_{f\in\mathcal{R}(f_0)} \Psi(f) \equiv \int_D f u_f dx = \int_D |\nabla u_f|^p dx, \tag{1.3}$$

where $f_0 \in L^\infty(D)$, and $\mathcal{R}(f_0)$ is the rearrangement class[1] generated by f_0 on D. In [12,16], the minimization problem (1.3) has been considered under restrict constraints imposed on the generator f_0; namely, that f_0 is strictly positive [16], and non-negative in [12]. These restrictions, on the one hand, turned out to be crucial in proving that the minimization problem (1.3) is *always* solvable, a result that is not guaranteed in the case where the generator is *indefinite* (f_0 takes on positive and negative values). On the other hand, the constraints mentioned above excludes many physical situations to fit the model described by the boundary value problem (1.1), especially when $p = 2$. Let us briefly explain this matter. In the case $p = 2$, (1.1) becomes:

$$\begin{cases} -\Delta u = f & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.4}$$

It is well known that (1.4) models the steady state of a vibrating membrane that occupies the region D, fixed at the boundary. The membrane is subject to a vertical force f. The function u denotes the displacement of the membrane from the rest position. The quantity $\int_D f u_f dx$, called total displacement, in a way measures the robustness of the membrane. So the minimization (1.3) seeks an optimal force, selected from all possible options in $\mathcal{R}(f_0)$, which minimizes the total displacement. In the present paper, we are allowing the vertical force f to be exerted in both directions *upwards* and *downwards*.

Another interpretation of (1.4) is related to the two dimensional ideal fluids. Let $\mathbf{U}$ denote the velocity field of a two dimensional ideal fluid occupying a region $\Omega \subset \mathbb{R}^3$ with cross section D. We assume the fluid is tangential to the boundary. Denoting the stream function by ζ, the relation between $\mathbf{U}$ and ζ is given by $\mathbf{U} = \nabla^\perp \zeta = \langle \partial\zeta/\partial y, -\partial\zeta/\partial x\rangle$. Thus, the vorticity function i.e. the z-component of the vorticity field $\nabla \times \mathbf{U} = \langle 0, 0, -\Delta\zeta\rangle$ is just $-\Delta\zeta$. Therefore, if we think of the functions u and f in (1.4) as the stream and vorticity functions, respectively, of a two dimensional ideal fluid, tangential to the boundary ($u = 0$ on ∂D), then the quantity $\int_D f u_f dx$ measures the potential energy of the fluid. Whence, the minimization (1.3) seeks an optimal vorticity function that minimizes the potential energy. In the case Δ is changed to Δ_p, the fluid becomes non-Newtonian. The main difference between a Newtonian and a non-Newtonian fluid is the viscosity (the ratio between the shear rate and shear stress). In case of Newtonian fluids, the viscosity is constant, hence the graph of the shear rate vs shear stress is a straight line that passes through the origin, but in the case of a non-Newtonian fluid the curve becomes either strictly concave (pseudoplastic) or strictly convex (Dilatant). The quantity to minimize, given in (1.3), we call the p-potential energy of the fluid.

The minimization problem (1.3), when $p = 2$, has been discussed in [6]. However, their technique fails to apply to the case of $p \neq 2$, simply because Δ is linear but Δ_p is not. Nonetheless, we still use the general theory of rearrangements that has been well established by G. R. Burton [4,5] as our guide map.

We show that when the generator is indefinite i.e. f_0 has both positive and negative parts, (1.3) will have no solutions in the rearrangement class, but will have one in the weak closure of it. Furthermore, we derive

[1] See Definition 2.2.