Nonlinear Analysis 145 (2016) 199–263

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

A quasiperiodically forced skew-product on the cylinder without fixed-curves

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ARTICLE INFO

Article history:
Received 31 March 2016
Accepted 21 August 2016
Communicated by Enzo Mitidieri

MSC:
primary 37C55
37C70

Keywords:
Quasiperiodically forced systems on the cylinder
Invariant strips

ABSTRACT

In Fabbri et al. (2005) the Sharkovskiĭ Theorem was extended to periodic orbits of strips of quasiperiodic skew products in the cylinder.

In this paper we deal with the following natural question that arises in this setting: *Does Sharkovskiĭ Theorem hold when restricted to curves instead of general strips?*

We answer this question in the negative by constructing a counterexample: We construct a map having a periodic orbit of period 2 of curves (which is, in fact, the upper and lower circles of the cylinder) and without any invariant curve.

In particular this shows that there exist quasiperiodic skew products in the cylinder without invariant curves.

1. Introduction

We consider the coexistence and implications between periodic objects of maps on the cylinder $\Omega = \mathbb{S}^1 \times \mathbb{I}$, of the form:

$$T\colon \begin{pmatrix} \theta \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} R_\omega(\theta) \\ \zeta(\theta, x) \end{pmatrix},$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $\mathbb{I}$ is a compact interval of the real line, $R_\omega(\theta) = \theta + \omega \pmod 1$ with $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $\zeta(\theta, x) = \zeta_\theta(x)$ is continuous on both variables. The class of all maps of the above type will be denoted by $\mathcal{S}(\Omega)$.

In this setting a very basic and natural question is the following: *is it true that any map in the class* $\mathcal{S}(\Omega)$ *has an invariant curve?*

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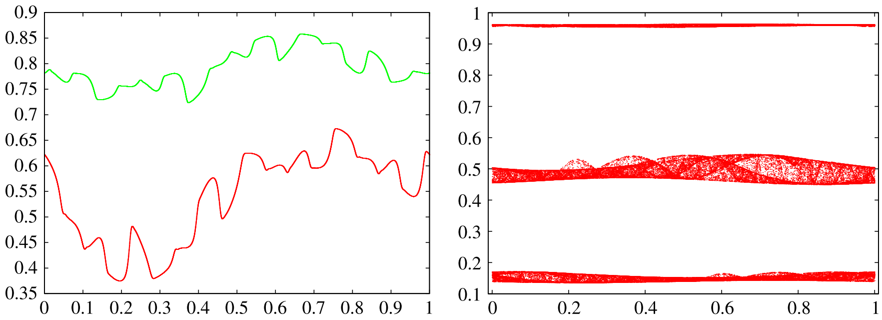
http://dx.doi.org/10.1016/j.na.2016.08.011



(a) $3.28x(1-x) + \frac{4}{100}\cos(2\pi\theta)$. (b) $3.85x(1-x)(1 + \frac{111}{10^5}\cos(2\pi\theta))$.

Fig. 1. In the left picture we show an example of two periodic orbit curves, and in the second we show a possible example of a three periodic orbit solid strips.

In [1], the authors created an appropriate topological framework that allowed them to obtain the following extension of the Sharkovskiĭ Theorem to the class $\mathcal{S}(\Omega)$.[1]

Let X be a compact metric space. We recall that a subset $G \subset X$ is *residual* if it contains the intersection of a countable family of open dense subsets in X.

In what follows, $\pi\colon \Omega \longrightarrow \mathbb{S}^1$ will denote the standard projection from Ω to the circle. Given a set $B \subset \mathbb{S}^1$, for convenience we will use the following notation:

$$\upuparrows B := \pi^{-1}(B) = B \times \mathbb{I} \subset \Omega.$$

In the particular case when $B = \{\theta\}$, instead of $\upuparrows\{\theta\}$ we will simply write $\upuparrows\theta$. Also, given $A \subset \Omega$, we will denote by $A^{\upuparrows B}$ the set

$$A \cap \upuparrows B = \{(\theta, x) \in \Omega \colon \theta \in B \text{ and } (\theta, x) \in A\}.$$

In the particular case when $B = \{\theta\}$, instead of $A^{\upuparrows\theta}$ we will simply write A^θ.

Instead of periodic points we use objects that project over the whole $\mathbb{S}^1$, called *strips* in [1, Definition 3.9]. A set $B \subset \Omega$ such that $\pi(B) = \mathbb{S}^1$ (i.e., B projects on the whole $\mathbb{S}^1$) will be called a *circular set.*

Definition 1.1. A *strip in* Ω is a compact circular set $B \subset \Omega$ such that B^θ is a closed interval (perhaps degenerate to a point) for every θ in a residual set of $\mathbb{S}^1$. ■

Given two strips A and B, we will write $A < B$ and $A \le B$ [1, Definition 3.13] if there exists a residual set $G \subset \mathbb{S}^1$, such that for every $(\theta, x) \in A^{\upuparrows G}$ and $(\theta, y) \in B^{\upuparrows G}$ it follows that $x < y$ and, respectively, $x \le y$. We say that the strips A and B are *ordered* (respectively *weakly ordered*) if either $A < B$ or $A > B$ (respectively $A \le B$ or $A \ge B$).

Definition 1.2 (*[1, Definition 3.15]*)**.** A strip $B \subset \Omega$ is called *n-periodic* for $F \in \mathcal{S}(\Omega)$ if $F^n(B) = B$ and the image sets $B,\ F(B),\ F^2(B), \dots, F^{n-1}(B)$ are pairwise disjoint and pairwise ordered (see Fig. 1 for examples). ■

[1] As already remarked in [1], instead of $\mathbb{S}^1$ we could take any compact metric space Θ that admits a minimal homeomorphism $R\colon \Theta \longrightarrow \Theta$ such that R^ℓ is minimal for every $\ell > 1$. However, for simplicity and clarity we will remain in the class $\mathcal{S}(\Omega)$.