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A measure-type Lagrange multiplier for the elastic–plastic torsion

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ABSTRACT

The existence of Lagrange multiplier as a Radon measure is ensured for an elastic–plastic torsion problem associated to a linear operator. The result is obtained by means of strong duality theory.

1. Introduction

The elastic–plastic torsion problem and its relationships with the obstacle problem have been deeply investigated in years 1965–1980. Later on these studies have been resumed, with particular regards to existence and properties of Lagrange multipliers.

According to von Mises [1] (see also [2,3]), the elastic–plastic torsion problem of a cylindrical bar with cross section Ω is to find a function $\psi(x)$ which vanishes on the boundary $\partial\Omega$ and, together with its first derivatives, is continuous on Ω; everywhere in Ω the gradient of ψ must have an absolute value (modulus) less than or equal to a given positive constant τ; whenever in Ω the strict inequality holds, the function ψ must satisfy the differential equation $\Delta\psi = -2\mu\theta$, where the positive constants μ and θ denote the shearing modulus and the angle of twist per unit length respectively.

In the planar case the existence and the properties of a smooth solution of the problem have been studied by Ting [3–5], formulating it as a variational minimum problem.

Multidimensional case has been studied in [6], where the author considers the following variational inequality: *Find* $u \in K_\nabla$ *such that*

$$\int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i}\left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx \geq \int_\Omega f(v-u)dx, \quad \forall v \in K_\nabla, \tag{1}$$

where

$$K_\nabla = \left\{ v \in H_0^{1,\infty}(\Omega) : \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \leq 1, \ \text{a.e. on } \Omega \right\},$$

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with Ω open, bounded subset of $\mathbb{R}^n$ either convex or with boundary of class $C^{1,1}$, $f = const. > 0$, and proves the existence of a Lagrange multiplier, namely, if u is the solution of variational inequality (1), then there exists a unique $\mu \in L^\infty(\Omega)$ such that:

$$\begin{cases} \mu \geq 0 \quad \text{a.e. in } \Omega \\ \mu(1 - |\nabla u|) = 0 \quad \text{a.e. in } \Omega \\ -\Delta u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mu \frac{\partial u}{\partial x_i}\right) = f \text{ in the sense of } D'(\Omega), \end{cases} \tag{2}$$

that is the solution of variational inequality (1) solves the elastic–plastic torsion problem. Conversely, if $u \in K_\nabla$ and there exists μ satisfying (2), then it is easily proved that u is the solution of (1).

The location and shape of the elastic and plastic regions $E = \{x \in \Omega : |\nabla u(x)| < 1\}$ and $P = \{x \in \Omega : |\nabla u(x)| = 1\}$ and the free boundary are studied in [7], while reinforcement problems are studied in [8].

Recently in [9] the authors prove the existence of Lagrange multipliers for the following variational inequality, more general than (1):

Find $u \in K_\nabla$ such that:

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx + \int_\Omega \lambda u(v-u)dx \geq \int_\Omega f(v-u)dx, \quad \forall v \in K_\nabla, \tag{3}$$

where λ and f are nonnegative constants.

In [10], the authors prove, under more general assumptions with respect to previous results, the existence of a Lagrange multiplier as a positive Radon measure.

Aim of the paper is to prove the existence of a Lagrange multiplier as a positive Radon measure for the following variational inequality more general than (1)

$$\int_\Omega \mathcal{L}u(v-u)\,dx \geq \int_\Omega f(v-u)\,dx, \quad \forall v \in K_\nabla, \tag{4}$$

with

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu$$

and $f \in L^p(\Omega)$, $p > 1$.

Note that variational inequality (4) cannot be reduced to a problem of minimum of a convex functional as considered in [10] and the assumptions are different too.

In the paper the existence of Lagrange multipliers as a Radon measure is ensured by means of strong duality theory in classical sense. This theory cannot be applied in order to obtain the existence of Lagrange multipliers as a L^2 function since in those settings the interior of the ordering cone, which defines the sign constraints, is empty (see [11,12]).

Using a very recent strong duality theory introduced in [13–15], the authors in [9] prove the existence of a L^∞ Lagrange multiplier for (4) assuming a constraint qualification condition (Assumption S), that is a necessary and sufficient condition in order that strong duality holds. In [16] the authors notice that such existence of a L^∞ Lagrange multiplier for (4) can be obtained in an easier way using Assumption S′, which is still equivalent to the strong duality. We refer also to [17], where an equivalence condition for the strong duality is given in a very general setting: with no condition on the map and constrains.

Finally variational problems with non-constant gradient constraints or with curl constraints have been studied in [18–20].

2. Main results

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain either convex or with boundary of class $C^{1,1}$. Let us consider the linear elliptic operator

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \tag{5}$$

with associated bilinear form on $H_0^{1,\infty}(\Omega) \times H_0^{1,\infty}(\Omega)$ given by

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} v + cuv\right) dx,$$