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Non self-similar blow-up solutions to the heat equation with nonlinear boundary conditions

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ABSTRACT

This paper is concerned with finite blow-up solutions of the heat equation with nonlinear boundary conditions. It is known that a rate of blow-up solutions is the same as the self-similar rate for a Sobolev subcritical case. A goal of this paper is to construct a blow-up solution whose blow-up rate is different from the self-similar rate for a Sobolev supercritical case.

1. Introduction

We study positive solutions of the heat equation with nonlinear boundary conditions:

$$\begin{cases} \partial_t u = \Delta u, & (x,t) \in \mathbb{R}^n_+ \times (0,T), \\ \partial_\nu u = u^q, & (x,t) \in \partial\mathbb{R}^n_+ \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n_+, \end{cases} \tag{1}$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}$, $\partial_\nu = -\partial/\partial x_n$, $q > 1$ and

$$u_0 \in C(\overline{\mathbb{R}^n_+}) \cap L^\infty(\mathbb{R}^n_+), \quad u_0(x) \geq 0.$$

It is well known that (1) admits a unique local classical solution $u(x,t) \in BC(\overline{\mathbb{R}^n_+} \times [0,\tau)) \cap C^{2,1}(\overline{\mathbb{R}^n_+} \times (0,\tau))$ for small $\tau > 0$, where $BC(\Omega) = C(\Omega) \cap L^\infty(\Omega)$. However by the presence of nonlinearity u^q on the boundary, a solution $u(x,t)$ may blow up in a finite time $T > 0$, namely

$$\limsup_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^n_+)} = \infty.$$

In fact, a solution of (1) actually blows up in a finite time under some conditions on the initial data (e.g. [1–3]). In this paper, we are concerned with the asymptotic behavior of blow-up solutions of (1). Let $q_S = n/(n-2)$ if $n \geq 3$ and $q_S = \infty$ if $n = 1, 2$. For the case $1 < q < q_S$, it is known that a finite time blow-up solution $u(x,t)$ of (1) satisfies

$$\sup_{t\in(0,T)} (T-t)^{1/2(q-1)} \|u(t)\|_{L^\infty(\mathbb{R}^n_+)} < \infty, \tag{2}$$

where $T > 0$ is the blow-up time of $u(x,t)$ [4,5]. More precisely, let $x_0 \in \partial\mathbb{R}^n_+$ be the blow-up point of $u(x,t)$, then the asymptotic behavior of $u(x,t)$ is described by the backward self-similar blow-up solution [6]:

$$\lim_{t\to T} \sup_{|z|<R(T-t)^{1/2}} \left| (T-t)^{1/2(q-1)} u(x_0+z,t) - \chi\left(z_n/\sqrt{T-t}\right) \right| \tag{3}$$

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for any $R > 0$, where $\chi(\xi)$ is a unique positive solution of

$$\begin{cases} \chi'' - \dfrac{\xi}{2}\chi' - \dfrac{\chi}{2(q-1)} = 0 & \text{for } \xi > 0, \\ \chi' = -\chi^q & \text{on } \xi = 0. \end{cases}$$

Following their works, more precise asymptotic behavior of blow-up solutions were studied in [7,8]. In general, the estimate (2) is known to be important as the first step to study the asymptotic behavior of blow-up solutions. Once (2) is derived, one may obtain more precise asymptotic behavior such as (3). However it is not yet known whether (2) always holds for the case $q \geq q_S$. An aim of this paper is to show the existence of finite time blow-up solutions of (1) which does not satisfy (2) for some range of $q \geq q_S$. This kind of non self-similar blow-up phenomenon was already studied in various semilinear parabolic equations. Particularly, this paper is motivated by [9,10]. In that paper, they studied finite time blow-up solutions of

$$u_t = \Delta u + u^p, \quad (x,t) \in \mathbb{R}^n \times (0,T). \tag{4}$$

There are vast papers devoting finite time blow-up solutions of (4) and their asymptotic behavior. Let

$$p_S = \begin{cases} \infty & \text{if } n = 1, 2, \\ \dfrac{n+2}{n-2} & \text{if } n \geq 3, \end{cases} \qquad p_{JL} = \begin{cases} \infty & \text{if } n \leq 10, \\ \dfrac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11. \end{cases}$$

As for the blow-up rate, it was shown in [11,12] that if $1 < p < p_S$, every finite blow-up solution of (4) satisfies

$$\sup_{t\in(0,T)} (T-t)^{1/(p-1)} \|u(t)\|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{5}$$

This estimate is corresponding to (2), which is called type I blow-up. However (5) does not hold in general for $p \geq p_S$. In fact, Herrero and Velázquez [9,10] constructed finite time blow-up solutions satisfying

$$\sup_{t\in(0,T)} (T-t)^{1/(p-1)} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = \infty$$

for $p > p_{JL}$ (see also [13]). This blow-up is called type II. They also gave the exact blow-up rate for type II blow-up solutions constructed in that paper. Their method relies on the matched asymptotic expansion technique. However this technique includes a formal argument, it is justified by Brouwer's fixed point type theorem with tough pointwise a priori estimates. This technique is known to be a strong tool to study the non self-similar phenomena in semilinear parabolic equations.

In this paper, following their arguments, we will construct non self-similar blow-up solutions of (1) which do not satisfy (2).

Theorem 1.1. *Let q be JL-supercritical (see Definition 3.1). Then there exists a positive x_n-axial symmetric initial data $u_0(x) \in BC(\overline{\mathbb{R}^n_+})$ such that a solution $u(x,t)$ of (1) with the initial data $u_0(x)$ blows up in a finite time $T > 0$ and satisfies*

$$\sup_{t\in(0,T)} (T-t)^{1/2(q-1)} \|u(t)\|_{L^\infty(\mathbb{R}^n_+)} = \infty. \tag{6}$$

Remark 1.1. We will construct blow-up solutions described in Theorem 1.1 with several exact blow-up rates. Their blow-up rates ($\|u(t)\|_{L^\infty(\mathbb{R}^n_+)} \sim (T-t)^{-p_i}$) are given by

$$p_i = \frac{m}{2} + \frac{m\lambda_{1\ell}}{\gamma - m}.$$

Undefined constants m, γ and $\lambda_{1\ell}$ are defined in contents of this paper.

Remark 1.2. As far as the author knows, this paper seems to be the first one which treats non self-similar blow-up solutions in a non radial setting. However we will see that our argument is reduced to a radial case in the matching process.

Our idea of the proof is almost the same as that of [10]. To study finite time blow-up solutions, we first introduce the self-similar variables as usual.

$$\varphi(y,s) = (T-t)^{1/2(q-1)} u\left((T-t)^{1/2}x, t\right), \quad T-t = e^{-s}.$$

Then we will construct a solution which converges to the singular stationary solution $U_\infty(y)$ in the self-similar variables. Since $U_\infty(0) = \infty$, this solution gives the desired non self-similar blow-up solution satisfying $\|\varphi(s)\|_\infty \to \infty$ as $s \to \infty$, which is equivalent to (6). To do that, we linearize the rescaled equation around the singular stationary solution $U_\infty(y)$ and construct a solution which behaves as

$$\varphi(y,s) \sim U_\infty(y) + ce^{-\lambda_\ell s}\phi_\ell(y), \tag{7}$$