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Traveling surface waves of moderate amplitude in shallow water

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ABSTRACT

We study traveling wave solutions of an equation for surface waves of moderate amplitude arising as a shallow water approximation of the Euler equations for inviscid, incompressible and homogeneous fluids. We obtain solitary waves of elevation and depression, including a family of solitary waves with compact support, where the amplitude may increase or decrease with respect to the wave speed. Our approach is based on techniques from dynamical systems and relies on a reformulation of the evolution equation as an autonomous Hamiltonian system which facilitates an explicit expression for bounded orbits in the phase plane to establish existence of the corresponding periodic and solitary traveling wave solutions.

1. Introduction and main result

A number of competing nonlinear model equations for water waves have been proposed to this day to account for fascinating phenomena, such as wave breaking or solitary waves, which are not captured by linear theory. The well-known Camassa–Holm equation [1] is one of the most prominent examples, due to its rich structural properties. It is an integrable infinite-dimensional Hamiltonian system [2–4] whose solitary waves are solitons [5,6]. Some of its classical solutions develop singularities in finite time in the form of wave breaking [7], and recover in the sense of global weak solutions after blow up [8,9]. For a discussion on integrability in the periodic case we refer the reader to [10,11], and a classification of weak traveling wave solutions of the Camassa–Holm equation may be found in [12]. The manifold of its enticing features led Johnson to demonstrate the relevance of the Camassa–Holm equation as a model for the propagation of shallow water waves of moderate amplitude. He proved that the horizontal component of the fluid velocity field at a certain depth within the fluid is indeed described by a Camassa–Holm equation [13,14]. Constantin and Lannes [15] followed up on the matter in search of a suitable corresponding equation for the free surface and derived an evolution equation for surface waves of moderate amplitude in the shallow water regime,

$$u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0. \tag{1}$$

The authors show that Eq. (1) approximates the governing equations to the same order as the Camassa–Holm equation, and also prove that the Cauchy problem on the line associated to (1), is locally well-posed [15]. Employing a semigroup approach due to Kato [16], Duruk [17] shows that this result also holds true for a larger class of initial data, as well as for the corresponding spatially periodic Cauchy problem [18]. Consequently, solutions of (1) depend continuously on their initial data in H^s for $s > 3/2$, and it can be shown that this dependence is not uniformly continuous [19]. In the context of Besov

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spaces, well-posedness is discussed [20] using Littlewood–Paley decomposition, along with a study about analytic solutions and persistence properties of strong solutions. One of the important aspects of Eq. (1) lies in its relevance for capturing the non-linear phenomenon of wave breaking [15,17], a feature it shares with the Camassa–Holm equation. While the latter equation is known to possess global solutions [9,21], it is not apparent how to obtain global control of the solutions of Eq. (1), owing to its involved structure and due to the higher order nonlinearities. However, passing to a moving frame one can study so-called traveling wave solutions, whose wave profiles move at constant speed in one direction without altering their shape. Introducing the traveling wave Ansatz

$$\xi = x - c\,t, \qquad u(\xi) = u(t, x), \tag{2}$$

Eq. (1) reads upon integration

$$(1 - c)u + 3u^2 - 2u^3 + 3u^4 + (1 + c + 14u)\ddot{u} + 7\dot{u} = E, \tag{3}$$

for some real constant E, where the dot denotes differentiation with respect to ξ. Existence of smooth solitary wave solutions of (3) which decay to zero at infinity has been established [22] for wave speeds $c > 1$, and their orbital stability has been deduced [23] employing an approach due to Grillakis, Shatah and Strauss [24] taking advantage of the Hamiltonian structure of (1). In the present paper we set out to improve the existence result [22] by loosening the assumption that solitary waves tend to zero at infinity. Allowing for a decay to an arbitrary constant, we establish existence of a variety of novel traveling wave solutions of (1).

Theorem 1.1. *For every speed $c \in \mathbb{R}\backslash\{c^*\}$ there exist peaked periodic, as well as smooth solitary and periodic traveling wave solutions of (1). Periodic waves may be obtained also for $c = c^*$, where $c^* \approx 0.35328$ is the unique real solution of $3c^3 + 30c^2 + 1031c - 368 = 0$.*

Moreover, the solitary waves can be characterized in terms of two parameters – the wave speed c and the level of the undisturbed water surface s – allowing us to determine the exact regions in this parameter space which give rise to the following types of waves, cf. Fig. 1:

- *Solitary waves in $\mathcal{C}^1$ with* compact support *on $\mathbb{R}$ (along the straight line $c + 1 + 14s = 0$).*
- *Smooth solitary waves of* elevation *whose amplitude is strictly* increasing *(in region* I*) or* decreasing *(in regions* II *and* III*) with respect to c, once E is fixed.*
- *Smooth solitary waves of* depression *whose amplitude is strictly* increasing *(in region* V*) or* decreasing *(in region* IV*) with respect to c, once E is fixed.*

All solitary waves are symmetric with respect to their unique crest/trough, they are monotonic and decay exponentially to the undisturbed water surface s at infinity.

The proof of these results hinges on the observation that for traveling waves, Eq. (1) may be written as an autonomous Hamiltonian system involving two parameters. This insight allows us to explicitly determine bounded orbits in the phase plane which correspond to solitary and periodic traveling waves of elevation as well as depression (Sections 2.1 and 2.2). Moreover, we characterize all solitary traveling waves in terms of two parameters—the wave speed c and the water level s of the undisturbed surface at infinity. This enables us to prove the existence a family of solitary waves with compact support (Section 2.3). Furthermore, we obtain a family of peaked periodic waves (Section 2.4). Our work also extends the analysis of qualitative properties regarding the shape of solitary waves given in [22]: we prove that the profile is strictly monotonic between crest and trough, and derive explicit algebraic curves in the parameter space (c, s) to determine the regions where their amplitude is increasing and decreasing with respect to the wave speed c once a value E is fixed (Section 3). These are quite remarkable properties of solitary waves which, to our knowledge, contribute novel aspects to the study of traveling waves in evolution equations for water waves. Our approach is in fact applicable to a large class of nonlinear dispersive equations, which we exemplify by a discussion of traveling waves of the aforementioned Camassa–Holm equation, including peaked continuous solitary waves (Section 4). Some of the more involved computations regarding the algebraic curves are carried out in the Appendix.

2. Existence of traveling waves

The proof of Theorem 1.1 relies on the fact that Eq. (1) has very nice structural features.

2.1. Hamiltonian formulation

Consider a general partial differential equation which, upon introducing the traveling wave Ansatz (2) can be transformed into an autonomous ordinary differential equation of the form

$$\ddot{u}(u - \bar{u}) + \frac{1}{2}(\dot{u})^2 + F'(u) = 0, \tag{4}$$