Nonlinear Analysis 102 (2014) 120–132

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Existence and concentration of ground states to a quasilinear problem with competing potentials ☆

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ARTICLE INFO

Article history:
Received 6 October 2013
Accepted 28 January 2014
Communicated by S. Carl

Keywords:
Quasilinear equation
Concentration
Competing potential
Nehari manifold

ABSTRACT

In this paper, we are concerned with the existence and concentration behavior of ground states for the following quasilinear problem with competing potentials

$$-\varepsilon^2\Delta u + V(x)u - \varepsilon^2\frac{1}{2}\Delta(u^2)u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u,$$

where $3 < q < p < 22^* - 1$, 2^* is the Sobolev critical exponent, $V(x)$ and $P(x)$ are positive and $Q(x)$ may be sign-changing. We show the existence of the ground states via the Nehari manifold method for $\varepsilon > 0$, and these ground states "concentrate" at a global minimum point of the least energy function $C(s)$ as $\varepsilon \to 0^+$.

1. Introduction

In this paper, we are concerned with the following quasilinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + W(x)\psi - \tilde{h}(|\psi|^2)\psi - \hbar^2\kappa\Delta\rho(|\psi|^2)\rho'(|\psi|^2)\psi, \tag{1.1}$$

where $\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $\hbar$ is the Planck constant, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a real constant, h and $\rho : \mathbb{R}^+ \to \mathbb{R}$ are real functions of essentially pure power forms. Such equations arise in various branches of mathematical physics and they have been the object of extensive study in recent years. When one consider the case where $\rho(s) = s$, $\kappa = \frac{1}{2}$ and look for the standing wave solutions of type $\psi = exp(-iEt/\hbar)u(x)$, then (1.1) leads to the following equation of elliptic type

$$-\varepsilon^2\Delta u + V(x)u - \varepsilon^2\frac{1}{2}\Delta(u^2)u = h(u), \tag{1.2}$$

where $\varepsilon = \hbar$, $V(x) = W(x) - E$ and $h(u) = \tilde{h}(|u|^2)u$.

One of the main difficulties of (1.2) is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for $N = 1$ (see [1]), since the quasilinear and non-convex term $-\frac{1}{2}\Delta(u^2)u$ appears. The first existence result involving variational methods due to [1] for the case where $\varepsilon = 1$, $N = 1$ or $V(x)$ is radially symmetrical for high dimensions by using a constrained minimization argument (see also [2] for the more general case). After then, there are some ideas and approaches were developed to overcome the difficulty. See [3] for a Nehari manifold argument. Transforming the quasilinear problems into semilinear problems by a change of variables is an effective way to deal with (1.2), see [4] for an Orlicz space framework and [5] for a Sobolev space frame. But this method does not work for the general

☆ Supported by NSFC (11101355 and 11361078), Key Project of Chinese Ministry of Education (212162) and NSFY of Yunnan Province (2011CI020) and China Scholarship Council.

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http://dx.doi.org/10.1016/j.na.2014.01.025
0362-546X/

quasilinear problems (see e.g. [6]). Recently, a perturbation method was developed in [7] (see also [6] for general quasilinear problems). The main idea is adding a regularizing term to recover the smoothness of the energy functional, so the standard minimax theory can be applied. Differ from the semilinear problems, another feather of the quasilinear problem (1.2) is that the critical exponent is not 2^* but 22^* (see [3]), where 2^* is the usual Sobolev exponent. The critical problems similar to (1.2) was considered in [8,9], see [6,10] for more general quasilinear critical problems. The existence of infinitely many solutions was obtained in [11,12] for the periodic quasilinear problems (in [12,13] the authors considered the more general case).

There are some recent works which considered the concentration property of solutions as $\varepsilon \to 0$. In [14], assuming the potential V is locally Hölder continuous, bounded from below by a positive constant and admitting a bounded domain $\Omega \subset \mathbb{R}^2$ with $\inf_\Omega V < \min_{\partial\Omega} V$, the authors showed that there are positive ground states for a 2-dimensional and critical problem concentrating around a local minimum point of V as $\varepsilon \to 0$ and with exponential decay. Similar results were proved in [15,16] for the case where dimension $N \geq 3$, and see [17] for the 1-dimensional case. The multiplicity of semiclassical solutions were established in [18] for the subcritical case and in [19] for the critical case.

In this paper, motivated by Wang and Zeng [20], we consider a quasilinear problem with competing potentials. More precisely, we are devoted to study the existence and concentration of positive ground states to the following quasilinear equation

$$-\varepsilon^2 \Delta u + V(x)u - \varepsilon^2 \frac{1}{2}\Delta(u^2)u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \qquad (P_\varepsilon)$$

where $3 < q < p < 22^* - 1$, V and P are continuous and positive functions, Q is a bounded and continuous function. A simple model of (P_ε) is the case of $Q = 0$, V has a global minimum and P has a global maximum, there is possibly a competition between V and P in the following sense: $V(x)$ would attract ground states to its minimum point but $P(x)$ would attract ground states to its maximum point. The competition will become more complex provided $Q \neq 0$, and this makes finding the concentration points become more delicate. To the best of our knowledge, there is no work concerning this case.

When $P(x) \equiv \lambda > 0$ and $Q \equiv 0$, a result in [4, Theorem 1.1] implies the existence of positive ground states of (P_ε) for any $\varepsilon > 0$. But this result cannot be applied directly to (P_ε) when $Q \neq 0$. However, the Nehari manifold is still well defined, even Q is sign-changing (see Lemma 2.1). So we follow [4] with some modifications to obtain a positive ground state for (P_ε) via the Nehari manifold method for $\varepsilon > 0$, and then we find that these ground states concentrate at a global minimum point of the least energy function $C(s)$ as $\varepsilon \to 0^+$ via a concentration-compactness argument similar to [20].

Throughout this paper, we always make the following assumptions:

(V) $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ and $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$.
(P) $P(x) \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$ is a positive function.
(Q) $Q(x) \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$.

To state our main result, we need two auxiliary problems. For each $s \in \mathbb{R}^N$, consider the following problems with parameter $s \in \mathbb{R}^N$

$$-\Delta u + V(s)u - \frac{1}{2}\Delta(u^2)u = P(s)|u|^{p-1}u + Q(s)|u|^{q-1}u. \qquad (P_s)$$

Denote the corresponding energy functional by I^s and the corresponding least energy by

$$C(s) := c(V(s), P(s), Q(s)) = \inf\{I^s(u) | u \text{ is a nontrivial solution of } (P_s)\}.$$

Also, we consider a "limit" problem

$$-\Delta u + V_\infty u - \frac{1}{2}\Delta(u^2)u = P_\infty|u|^{p-1}u + Q_\infty|u|^{q-1}u, \qquad (P_\infty)$$

and denote the corresponding least energy by $c_\infty = c(V_\infty, P_\infty, Q_\infty)$, where

$$V_\infty := \liminf_{|x|\to\infty} V(x), \qquad P_\infty := \limsup_{|x|\to\infty} P(x) \quad \text{and} \quad Q_\infty := \limsup_{|x|\to\infty} Q(x).$$

Our main result is the following.

Theorem 1.1. *Suppose that (V), (P) and (Q) are satisfied, $3 < q < p < 22^* - 1$. If*

$$\inf_{s\in\mathbb{R}^N} C(s) < c_\infty, \qquad (1.6)$$

then there is an $\varepsilon_0 > 0$ such that

(1) *(P_ε) possesses a positive ground state solution u_ε for all $\varepsilon \in (0, \varepsilon_0)$.*
(2) *u_ε possesses at most one local (hence global) maximum point x_ε in $\mathbb{R}^N$ such that*

$$\lim_{\varepsilon\to 0^+} C(x_\varepsilon) = \inf_{s\in\mathbb{R}^N} C(s).$$

(3) *there exist $C_1, C_2 > 0$ such that*

$$u_\varepsilon(x) \leq C_1 e^{-C_2|\frac{x-x_\varepsilon}{\varepsilon}|}.$$