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On the superlinear problems involving the $p(x)$-Laplacian and a non-local term without AR-condition☆

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ABSTRACT

In this paper, we consider a kind of nonlinear eigenvalue problem without Ambrosetti–Rabinowitz condition, that is,

$$\begin{cases} -\eta[u]\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), & \text{a.e. in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathrm{R}^N$ is a bounded domain, $p : \overline{\Omega} \to (1, +\infty)$ is a continuous function, $\eta[u]$ is a non-local term defined by the following relation

$$\eta[u] = 2 + \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^{\frac{p^+}{p^-}-1} + \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^{\frac{p^-}{p^+}-1}.$$

Here $\lambda > 0$ is a parameter, and $f(x,u)$ is $\frac{(p^+)^2}{p^-}$-superlinear at infinity. Existence of nontrivial solution is established for arbitrary $\lambda > 0$. We first prove the existence of nontrivial solutions of the system for almost every parameter $\lambda > 0$ by using Mountain Pass Theorem, and then we consider the continuation of the solutions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following nonlinear eigenvalue problem involving the $p(x)$-Laplacian:

$$\begin{cases} -\eta[u]\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), & \text{a.e. in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here $p : \overline{\Omega} \to (1, +\infty)$ is a continuous function, $\eta[u]$ is a non-local term defined by the following relation

$$\eta[u] = 2 + \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^{\frac{p^+}{p^-}-1} + \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^{\frac{p^-}{p^+}-1},$$

and $f \in C(\overline{\Omega} \times \mathbb{R})$ is superlinear and do not satisfy Ambrosetti and Rabinowitz type growth condition, $\lambda > 0$ is a parameter.

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Mihăilescu and Stancu-Dumitru in [1] established an existence of nontrivial solution for problem (P), by assuming the following conditions:

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition and

$$|f(x,u)| \le c(1+|u|^{\alpha(x)-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C_+(\overline{\Omega})$ and $\frac{(p^+)^2}{p^-} < \alpha^- \le \alpha^+ < \frac{Np^-}{N-p^-}$.

(f_2) there exist $\theta > \frac{(p^+)^2}{p^-}$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) \le f(x,t)t, \quad \forall x \in \overline{\Omega},\ t \in \mathbb{R} \text{ with } |t| \ge t_0,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(f_3) for any $\lambda \in (0, \lambda_1)$ the following relation holds true

$$\frac{\lambda f(x,t)}{|t|^{p(x)-2}t} = \lambda < \lambda_1,$$

for any $t \in \mathbb{R}$ satisfying $|t| \le 1$ and $x \in \Omega$.

For the case, when $p(x) \equiv 2$, the problem (P) has been studied extensively by many researchers, who tried to drop the above condition (f_2), see for instance, [2–6]. It is well known that, the condition (f_2) is quite important in the sense that, it not only ensures that the Euler–Lagrange functional associated with problem (P) has a mountain pass geometry, but also guarantees that the Palais–Smale sequence of the Euler–Lagrange functional is bounded. However, this condition is very strong, and thus undoubtedly eliminates many nonlinearities. We recall that (f_2) implies another weaker condition

$$|F(x,t)| \ge c_1|t|^\theta - c_2, \quad c_1, c_2 > 0, x \in \Omega, t \in \mathbb{R} \text{ and } \theta > \frac{(p^+)^2}{p^-}.$$

This says that $f(x,t)$ is $\frac{(p^+)^2}{p^-}$-superlinear at infinity in the sense that

(f_5) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^\tau} = +\infty$ with $\tau = \frac{(p^+)^2}{p^-}$, uniformly a.e. $x \in \Omega$.

In [1], the author considered problem (P) with a particular nonlinearity

$$f(x,t) = \begin{cases} |t|^{p(x)-2}t, & |t| < 1, \\ |t|^{r(x)-2}t, & |t| \ge 1. \end{cases}$$

This means that $f(x,t)$ satisfies Ambrosetti and Rabinowitz type growth condition ((A–R) type conditions for short). Using the mountain-pass theorem of Ambrosetti and Rabinowitz, the author obtained the existence of a continuous family of eigenvalue lying in a neighborhood at the right of the origin.

In this paper, we consider problem (P) in the case when the nonlinear term $f(x,t)$ is $\frac{(p^+)^2}{p^-}$-superlinear at infinity but does not satisfy the (A–R) type condition (f_2) as in [1]. More precisely, we assume that $f(x,t)$ satisfies the following general conditions:

(h_1) $f : \Omega \times \mathrm{R} \to \mathrm{R}$ satisfies Caratheodory condition and

$$|f(x,u)| \le c_3 + c_4|u|^{\alpha(x)-1}, \quad \forall (x,t) \in \Omega \times \mathrm{R},$$

where $\alpha \in C_+(\overline{\Omega})$ and $p^+ < \alpha^- \le \alpha(x) < p^*(x)$.

(h_2) the following limit holds uniformly for a.e. $x \in \Omega$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^\theta} = +\infty, \quad \text{where } \theta = \frac{(p^+)^2}{p^-}.$$

(h_3) $f(x,t) = o(|t|^{p(x)-2}t)$, $t \to 0$, for $x \in \Omega$ uniformly.

(h_4) there exists $t_0 > 0$ such that for $\forall x \in \Omega$,

$$\frac{f(x,t)}{t^{p^+-1}} \text{ is increasing in } t \ge t_0 \text{ and decreasing in } t \le -t_0.$$

We remark that the function $f(x,t) = t^{\alpha(x)-1}(\alpha(x)\ln t + 1)$ satisfies condition (h_4), but it does not satisfy (f_2) if $2\alpha^- > \frac{(p^+)^2}{p^-} > \alpha^+$. Moreover, the condition (h_4) implies another much weaker condition, that is,

(h_4') There is $C_* > 0$, such that

$$tf(x,t) - p^+F(x,t) \le sf(x,s) - p^+F(x,s) + C_*$$

for all $0 < t < s$ or $s < t < 0$.

Following along the same lines as in [6], we can obtain the existence of the nontrivial weak solution for all $\lambda > 0$.

The remainder of the paper is organized as follows. In Section 2, we will recall the definitions and some properties of variable exponent Sobolev spaces. In Section 3 we will state and prove the main results of the paper.