Nonlinear Analysis 102 (2014) 220–225

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A gap theorem for Willmore tori and an application to the Willmore flow

Andrea Mondino [a,*], Huy The Nguyen [b]

[a] ETH, Rämistrasse 101, Zurich, Switzerland

[b] School of Mathematics and Physics, The University of Queensland, St Lucia, Brisbane, 4072, Australia

ARTICLE INFO

Article history:
Received 20 August 2013
Accepted 20 February 2014
Communicated by Enzo Mitidieri

Keywords:
Geometric analysis
Willmore functional
Surface theory
Nonlinear elliptic equation of 4th order
Geometric flows

ABSTRACT

In 1965, Willmore conjectured that the integral of the square of the mean curvature of a torus immersed in $\mathbb{R}^3$ is at least $2\pi^2$ and attains this minimal value if and only if the torus is a Möbius transform of the Clifford torus. This was recently proved by Marques and Neves (2012). In this paper, we show for tori that there is a gap to the next critical point of the Willmore energy and we discuss an application to the Willmore flow. We also prove an energy gap from the Clifford torus to surfaces of higher genus.

1. Introduction

Let Σ be a compact Riemann surface and $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion. Then the Willmore energy is defined to be

$$\mathcal{W}(f) = \int |H|^2 d\mu_g$$

where g is the induced metric, $d\mu_g$ is the induced area form and H is the mean curvature (we adopt the convention that H is half of the trace of the second fundamental form). It is well known that the Willmore energy is invariant under conformal transformations of $\mathbb{R}^3$, the so called Möbius transformations. It was shown by Willmore [1] that, for surfaces in $\mathbb{R}^3$, the Willmore energy satisfies the inequality $\mathcal{W}(f) \geq 4\pi$ with equality if and only if the surface is a round sphere. He conjectured also that every torus satisfies the inequality $\mathcal{W}(f) \geq 2\pi^2$ with equality if and only if the surface is a Möbius transform of the Clifford torus, what we will call *conformal Clifford torus*. This conjecture was recently proved by Marques and Neves [2].

A natural question is the existence and energy values of non-minimizing critical points of the Willmore energy, the so called Willmore surfaces. Note that the Willmore energy is conformally invariant, therefore inversions of complete non-compact minimal surfaces with appropriate growth at infinity are Willmore surfaces. It was proved by Bryant [3] that smooth Willmore surfaces in $\mathbb{R}^3$ that are topologically spheres are all inversions of complete non-compact minimal spheres with embedded planar ends. Using the Weierstrass–Enneper representation, in principle, all these minimal surfaces may be classified. In particular, the Willmore energies of these surfaces are quantized, $\mathcal{W}(f) = 4\pi k,\ k \geq 1$ and the first non-trivial value is $\mathcal{W}(f) = 16\pi$. Hence the gap to the next critical value of the Willmore energy among spheres is 12π. The values $k = 2, 3$ were recently investigated by Lamm and Nguyen [4] and it was shown that they correspond to inversions of catenoids, Enneper's minimal surface or trinoids. These surfaces are not smooth but have point singularities or branch points.

* Corresponding author. Tel.: +41 393292040701.
E-mail addresses: andrea.mondino@math.ethz.ch, a_mondino@hotmail.com (A. Mondino), huy.nguyen@maths.uq.edu.au (H.T. Nguyen).

http://dx.doi.org/10.1016/j.na.2014.02.015
0362-546X/

While, as just described, the family of Willmore spheres is quite well understood, much less is known for Willmore tori. By the work of Pinkall [5], it is known the existence of infinitely many Willmore tori but it is an open problem whether or not the Willmore energies attained by critical tori are isolated. In this paper, we will show that there exists a gap from the minimizing conformal Clifford torus to the next critical point of the Willmore energy, namely we prove the following result.

Theorem 1.1 (*Gap Theorem for Willmore Tori*)**.** *There exists $\epsilon_0 > 0$ such that, if $T \subset \mathbb{R}^3$ is a smoothly immersed Willmore torus (i.e. a critical point for the Willmore functional) with*

$$\mathcal{W}(T) \leq 2\pi^2 + \epsilon_0, \tag{1.1}$$

then T is the image, under a Möbius transformation of $\mathbb{R}^3$, of the standard Clifford torus T_{Cl}; in particular $\mathcal{W}(T) = 2\pi^2$.

Remark 1.2. The analogous gap result for spheres in codimension one follows by the aforementioned work of Bryant [3], in codimension two was obtained by Montiel [6], and in arbitrary codimension was proved by Kuwert and Schätzle [7] and subsequently by Bernard and Rivière [8].

By the work of Chill–Fasangova–Schätzle [9] (which is a continuation of previous works on the Willmore flow by Kuwert and Schätzle [7,10]) the gap Theorem 1.1 has the following application to the Willmore flow, i.e. the L^2-gradient flow of the Willmore functional:

Corollary 1.3. *Let $f_C : T \mapsto \mathbb{R}^3$ be a conformal Clifford torus. i.e. a regular Möbius transformation of the Clifford torus T_{Cl}.*

There exists $\varepsilon_0 > 0$ such that if $\|f_0 - f_C\|_{W^{2,2}\cap C^1} \leq \varepsilon_0$ then, after reparametrization by appropriate diffeomorphisms $\Psi_t : T \to T$, the Willmore flow $(f_t)_t$ with initial data f_0 exists globally and converges smoothly to a conformal Clifford torus $\tilde{f}_C$, that is

$$f_t \circ \Psi_t \to \tilde{f}_C, \quad \text{as } t \to \infty.$$

We finish the paper by showing the following gap theorem from the Clifford torus to higher genus (even non-Willmore) surfaces. The proof is a combination of the large genus limit of the Willmore energy proved by Kuwert–Li–Schätzle [11] and the proof of the Willmore conjecture by Marques and Neves [2].

Theorem 1.4. *There exists $\varepsilon_0 > 0$ such that if $\Sigma \subset \mathbb{R}^3$ is a smooth immersed surface of genus at least two, then*

$$\mathcal{W}(\Sigma) \geq 2\pi^2 + \varepsilon_0.$$

The paper is organized as follows: in Section 2 we gather together the necessary material and definitions that we will require in the paper. In Section 3 we prove the gap Theorem 1.1 for Willmore tori and, in Section 4, we use it to show the convergence of the Willmore flow of tori that are sufficiently close to the conformal Clifford torus, namely we prove Corollary 1.3. Finally, Section 5 is devoted to the proof of the energy gap for higher genus surfaces, namely Theorem 1.4.

2. Preliminaries

Throughout the paper T_{Cl} will denote the standard Clifford torus embedded in $\mathbb{R}^3$ (i.e. the torus obtained by revolution around the z-axis of a unit circle in the xz-plane with center at $(\sqrt{2}, 0, 0)$) and $\mathcal{M}$ will denote the Möbius group of $\mathbb{R}^3$.

Recall from the classical paper of Weiner (see in particular Lemma 3.3, Proposition 3.1 and Corollary 1 in [12]), that the second differential of the Willmore functional $\mathcal{W}''$ on the Clifford torus defines a positive semidefinite bounded symmetric bilinear form on $H^2(T_{\mathrm{Cl}})$, the Sobolev space of L^2 integrable functions with first and second weak derivatives in L^2; in formulas, the second differential of the Willmore functional pulled back on S^3 via stereographic projection on the standard Clifford torus $\frac{1}{\sqrt{2}}(S^1 \times S^1) \subset S^3 \subset \mathbb{R}^4$ is given by

$$\mathcal{W}''(u, v) := \int (\Delta + |A|^2)[u] \cdot (\Delta + |A|^2 + 2)[v]\, d\mu_g,$$

where Δ is the Laplace–Beltrami operator on the Clifford torus, $|A| \equiv 2$ is the norm of the second fundamental form, and the integral is computed with respect to the surface measure $d\mu_g$.

Moreover the kernel of the bilinear form consists of infinitesimal Möbius transformations:

$$K := \mathrm{Ker}(\mathcal{W}'') = \{\text{infinitesimal Möbius transformations on } T_{\mathrm{Cl}}\} \subset C^\infty(T_{\mathrm{Cl}}), \tag{2.1}$$

i.e. for every $w \in K$ there exists a Möbius transformation $\Phi_w \in \mathcal{M}$ such that for $t \in \mathbb{R}$ small enough

$$\mathrm{Exp}_{T_{\mathrm{Cl}}}(tw) = \Phi_{tw}(T_{\mathrm{Cl}}),$$

where $\mathrm{Exp}_{T_{\mathrm{Cl}}}(tw)$ is the exponential in the normal direction with base surface T_{Cl} and of magnitude $tw \in C^\infty(T_{\mathrm{Cl}})$.

Again from [12], called $K^\perp \subset H^2(T_{\mathrm{Cl}})$ the orthogonal space to K in H^2, one also has that $\mathcal{W}''|_{K^\perp}$ is positive definite and defines a scalar product on $K^\perp$ equivalent to the restriction to $K^\perp$ of the $H^2(T_{\mathrm{Cl}})$ scalar product:

$$\mathcal{W}''(w, w) \geq \lambda \|w\|^2_{H^2(T_{\mathrm{Cl}})} \quad \forall\, w \in K^\perp, \tag{2.2}$$

for some $\lambda > 0$.