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Large time behavior for a porous medium equation in a nonhomogeneous medium with critical density

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ABSTRACT

We study the large time behavior of solutions to the Cauchy problem for the porous medium equation in nonhomogeneous media with critical singular density

$$|x|^{-2}\partial_t u = \Delta u^m, \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

where $m > 1$ and $N \geq 3$, with nonnegative initial condition $u(x,0) = u_0(x) \geq 0$. The asymptotic behavior proves to have some interesting and striking properties. We show that there are different asymptotic profiles for the solutions, depending on whether the continuous initial data u_0 vanishes at $x = 0$ or not. Moreover, when $u_0(0) = 0$, we show the convergence towards a peak-type profile presenting a jump discontinuity, coming from an interesting asymptotic simplification to a conservation law, while when $u_0(0) > 0$, the limit profile remains continuous. These phenomena illustrate the strong effect of the singularity at $x = 0$. We improve the time scale of the convergence in sets avoiding the singularity. On the way, we also study the large-time behavior for a porous medium equation with convection which is interesting for itself.

1. Introduction

The goal of this paper is to study the asymptotic behavior of solutions to the Cauchy problem for the following porous medium equation in nonhomogeneous media with critical singular density:

$$|x|^{-2}\partial_t u(x,t) = \Delta u^m(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

with $m > 1$ and $N \geq 3$, with nonnegative initial condition

$$u(x,0) = u_0(x) \geq 0. \tag{1.2}$$

An important feature of this equation is the influence of the density that is at the same time singular at $x = 0$ and degenerate at infinity, giving rise to very interesting and unexpected results.

Equations of type (1.1) with general densities, more precisely

$$\varrho(x)\partial_t u(x,t) = \Delta u^m(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.3}$$

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where ϱ is a density function with suitable behavior, have been proposed by Kamin and Rosenau in a series of papers [1–3] to model thermal propagation by radiation in non-homogeneous plasma. Afterwards, a huge development of the mathematical theory associated to Eq. (1.3) begun, usually under conditions such as

$$\varrho(x) \sim |x|^{-\gamma}, \quad \text{as } |x| \to \infty,$$

for some $\gamma > 0$, as for example in the following papers [4–10] where its qualitative properties and asymptotic behavior are studied. In particular, along these references, the basic existence and regularity properties are proved under suitable conditions for the initial data, and a detailed study for the asymptotic behavior for $\gamma \neq 2$ has been done. Thus, it has been noticed that for $\gamma \in (0, 2)$, the solutions have similar qualitative properties to the ones of the standard porous medium equation

$$u_t = \Delta u^m, \tag{1.4}$$

see [6,9], while for $\gamma > 2$ they are quite different [10]. Thus, the value $\gamma = 2$ is critical. A first step in the study of Eq. (1.3) with $\varrho(x) \sim |x|^{-2}$ at infinity, but ϱ regular at $x = 0$, has been done in the recent paper [11], having as starting point some conjectures and comments in [10].

On the other hand, concerning the asymptotic behavior of solutions to Eq. (1.3), it is shown that the profiles are special solutions of Eq. (1.1), giving thus rise to the natural problem of the study of the pure power density case $\varrho(x) = |x|^{-\gamma}$. A special feature of Eq. (1.1), besides its general interest for classifying asymptotic profiles for the general case (1.3), is the fact that a strong singularity appears at $x = 0$. The presence of this singular coefficient (in contrast with the above mentioned papers where $\varrho(x)$ is supposed regular at $x = 0$), introduces some unexpected mathematical phenomena, as the appearance of different profiles for different initial data only near $x = 0$, and backward evolution of the profiles.

Recently, in a previous work [12], the authors proved some of these interesting and striking features for the easier case of the linear equation

$$|x|^{-2}\partial_t u(x,t) = \Delta u, \quad (x,t) \in \mathbb{R}^N \times (0, \infty), \tag{1.5}$$

where all the profiles are explicit and one can use the theory of the heat equation. Moreover, a formal study of the radially symmetric solutions to Eq. (1.1) for general γ has been performed in [13], including some mappings that will be useful in the sequel for the case $\gamma = 2$.

Before stating and explaining our main results, we want to mention that we only consider dimensions $N \geq 3$, letting apart the cases $N = 1$ and $N = 2$ for a further work, due to some differences in the method.

Main results. In the present paper, we deal with the Cauchy problem associated to Eq. (1.1) with initial condition

$$u_0(x) := u(x, 0) \in L^1_2(\mathbb{R}^N), \quad u_0 \geq 0, \tag{1.6}$$

where $N \geq 3$ and, as usual,

$$L^1_2(\mathbb{R}^N) := \left\{ h : \mathbb{R}^N \mapsto \mathbb{R},\ h \text{ measurable}, \int_{\mathbb{R}^N} |x|^{-2} h(x)\,dx < \infty \right\}.$$

In Section 2 we make a review of the notion of *strong* solution to (1.1) and the well-posedness results we need. In particular, for any initial condition u_0 as above, there exists a strong solution u having some additional qualitative properties, see Theorem 2.2.

As it will become clear from the subsequent analysis, there is a branching point for the large-time behavior of general solutions to (1.1). Similar to the linear case $m = 1$ studied in our previous paper [12], a big difference is related to whether $u_0(0) = 0$ or $u_0(0) \neq 0$. Let us state rigorously our results.

1. *Initial data u_0 such that $u_0(0) = 0$.* Let us introduce the following weighted norm for $1 \leq p < \infty$ and a measurable function $h : \mathbb{R}^N \mapsto \mathbb{R}$:

$$\|h\|_{p,N} = \left(\int_{\mathbb{R}^N} |x|^{-N} |h(x)|^p\,dx \right)^{1/p}.$$

We state first a simple convergence result in the previous integral norm, which is interesting by itself but can be also seen as a preliminary.

Theorem 1.1. *Let u be a* radially symmetric *solution to Eq. (1.1) with initial condition u_0 satisfying (1.6) and furthermore*

$$u_0(0) = 0, \qquad M_{u_0} := \int_{\mathbb{R}^N} |x|^{-N} u_0(x)\,dx < \infty. \tag{1.7}$$

Then, for any $1 \leq p < \infty$, we have

$$\lim_{t\to\infty} t^{(p-1)/mp} \|u(x,t) - F(x,t)\|_{p,N} = 0, \tag{1.8}$$