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Decay and growth estimates for solutions of second-order and third-order differential-operator equations

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ABSTRACT

We obtained decay and growth estimates for solutions of second-order and third-order differential-operator equations in a Hilbert space. Applications to initial–boundary value problems for linear and nonlinear non-stationary partial differential equations modeling the strongly damped nonlinear improved Boussinesq equation, the dual-phase-lag heat conduction equations, the equation describing wave propagation in relaxing media, and the Moore–Gibson–Thompson equation are given.

1. Introduction

We consider, in a Hilbert space H equipped with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$, a second-order differential-operator equation of the form

$$Pu_{tt} + Qu_t + G(u) = 0 \tag{1.1}$$

and a third-order linear differential-operator equation of the form

$$u_{ttt} + \mathcal{A}u_{tt} + \mathcal{B}u_t + \mathcal{C}u = 0. \tag{1.2}$$

Here P, Q, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are linear, positive and self-adjoint operators and G is a gradient operator with potential $\mathcal{G}$. We assume that

$$D(G) \subseteq D(Q), \qquad \mathcal{G}(u) \geq 0, \quad \forall u \in D(G) \tag{1.3}$$

and

$$(G(u), u) - \mathcal{G}(u) \geq k_0 \|Q^{\frac{1}{2}} u\|^2, \quad \forall u \in D(G), \tag{1.4}$$

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where k_0 is some positive number. Clearly G might be a linear positive operator too. We assume that the domains of definition of the operators $D(\mathcal{A}), D(\mathcal{B}), D(\mathcal{C})$ and $D(G)$ are dense linear subspaces of H. We study strong solutions of Eqs. (1.1) and (1.2), i.e. solutions for which all terms involved by the corresponding equation belong to $L^2(0, T; H)$ for each $T > 0$.

The problem of global stability of solutions to differential-operator equations is inspired by problems of global stability of solutions to the Cauchy problem and initial–boundary value problems for various dissipative evolutionary partial differential equations. There are many publications devoted to the global stability of solutions of linear and nonlinear partial differential equations of second order in time, where exponential decay estimates are also obtained (see [1–9] and references therein).

In contrast to the case for work on the stability and instability of solutions to second-order equations, we know of just a few results on the global stability of solutions and results on decay estimates for evolutionary partial differential equations of third order in time [10–16].

2. The second-order equation

First we obtain an exponential decay estimate for solutions to Eq. (1.1) under the following additional condition on the operators P and Q:

$$D(Q) \subseteq D(P), \qquad d_0\|P^{\frac{1}{2}}u\|^2 \leq \|Q^{\frac{1}{2}}u\|^2, \quad \forall u \in D(Q), d_0 > 0. \tag{2.5}$$

Multiplication of (1.1) by $u_t + \varepsilon u$ gives the relation

$$\frac{d}{dt}\left[\|P^{\frac{1}{2}}u_t(t)\|^2 + 2\mathcal{G}(u(t)) + 2\varepsilon(Pu_t(t), u(t)) + \varepsilon\|Q^{\frac{1}{2}}u(t)\|^2\right]$$
$$+ 2\|Q^{\frac{1}{2}}u_t(t)\|^2 + 2\varepsilon(G(u(t), u(t))) - 2\varepsilon\|P^{\frac{1}{2}}u_t(t)\|^2 = 0. \tag{2.6}$$

By using the conditions (1.4) and (2.5), we obtain from (2.6)

$$\frac{d}{dt}\left[\|P^{\frac{1}{2}}u_t(t)\|^2 + 2\mathcal{G}(u(t)) + 2\varepsilon(Pu_t(t), u(t)) + \varepsilon\|Q^{\frac{1}{2}}u(t)\|^2\right]$$
$$+ 2(1 - \varepsilon d_0^{-1})\|Q^{\frac{1}{2}}u_t(t)\|^2 + 2\varepsilon\mathcal{G}(u(t)) + 2\varepsilon k_0\|Q^{\frac{1}{2}}u(t)\|^2 \leq 0. \tag{2.7}$$

Due to the Schwarz inequality and the condition (2.5) we have

$$2\varepsilon|(Pu_t, u)| \leq 2\varepsilon\|P^{\frac{1}{2}}u_t\|\|P^{\frac{1}{2}}u\| \leq \frac{1}{2}\|P^{\frac{1}{2}}u_t\|^2 + 2\varepsilon^2 d_0^{-1}\|Q^{\frac{1}{2}}u\|.$$

Thus, we choose in the last inequality $\varepsilon = \frac{d_0}{4}$ and obtain

$$\frac{d}{dt}L(u(t)) + \frac{3}{2}\|Q^{\frac{1}{2}}u_t(t)\|^2 + \frac{1}{2}d_0\mathcal{G}(u(t)) + \frac{d_0k_0}{2}\|Q^{\frac{1}{2}}u(t)\|^2 \leq 0, \tag{2.8}$$

where

$$L(u(t)) := \|P^{\frac{1}{2}}u_t(t)\|^2 + 2\mathcal{G}(u(t)) + \frac{d_0}{2}(Pu_t(t), u(t)) + \frac{d_0}{4}\|Q^{\frac{1}{2}}u(t)\|^2$$
$$\geq \frac{1}{2}\|P^{\frac{1}{2}}u_t(t)\|^2 + 2\mathcal{G}(u(t)) + \frac{d_0}{8}\|Q^{\frac{1}{2}}u(t)\|^2. \tag{2.9}$$

Adding to the left hand side of (2.8) the expression $\delta L(u(t)) - \delta L(u(t))$ we obtain

$$\frac{d}{dt}L(u(t)) + \delta L(u(t)) + \frac{3}{2}\|Q^{\frac{1}{2}}u_t(t)\|^2 + \left(\frac{1}{2}d_0 - 2\delta\right)\mathcal{G}(u(t))$$
$$+ \frac{d_0k_0}{2}\|Q^{\frac{1}{2}}u(t)\|^2 - \delta\|P^{\frac{1}{2}}u_t(t)\|^2 - \delta\frac{d_0}{2}(Pu_t(t), u(t)) \leq 0. \tag{2.10}$$

Here $\delta > 0$ is a parameter which we will choose below. Due to the condition (2.5) and the Schwarz inequality we have

$$\delta\|P^{\frac{1}{2}}u_t(t)\|^2 \leq d_0^{-1}\delta\|Q^{\frac{1}{2}}u_t(t)\|^2, \qquad \delta\frac{d_0}{2}(Pu_t(t), u(t)) \leq \frac{\delta}{4}\|Q^{\frac{1}{2}}u_t(t)\|^2 + \frac{\delta}{4}\|Q^{\frac{1}{2}}u(t)\|^2.$$

Employing the last inequalities we obtain from (2.10) the inequality

$$\frac{d}{dt}L(u(t)) + \delta L(u(t)) + \left(\frac{3}{2} - \frac{\delta}{d_0} - \frac{\delta}{4}\right)\|Q^{\frac{1}{2}}u_t(t)\|^2$$
$$+ \left(\frac{1}{2}d_0 - 2\delta\right)\mathcal{G}(u(t)) + \left(\frac{d_0k_0}{2} - \frac{\delta}{4}\right)\|Q^{\frac{1}{2}}u(t)\|^2 \leq 0. \tag{2.11}$$