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Asymptotic behavior for the heat equation in nonhomogeneous media with critical density

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ABSTRACT

We study the long-time behavior of solutions to the heat equation in nonhomogeneous media with critical singular density

$$|x|^{-2}\partial_t u = \Delta u, \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

in dimensions $N \geq 3$. The asymptotic behavior proves to have some interesting and quite striking properties. We show that there are two completely different asymptotic profiles depending on whether the initial data u_0 vanishes at $x = 0$ or not. Moreover, in the former the results are true only for radially symmetric solutions, and we provide counterexamples to convergence to symmetric profiles in the general case.

1. Introduction

The aim of this work is to establish the asymptotic behavior of solutions to the following heat equation in nonhomogeneous media with critical density:

$$|x|^{-2}\partial_t u(x,t) = \Delta u(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty) \tag{1.1}$$

as a part of an ongoing project of studying the asymptotic behavior for

$$|x|^{-2}\partial_t u(x,t) = \Delta u^m(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.2}$$

with $m \geq 1$. The technically more involved problem with the large-time behavior for $m > 1$ will be studied in a forthcoming paper [1].

Equations of type (1.2) with general densities were proposed by Kamin and Rosenau in a series of papers [2–4] to model thermal propagation by radiation in nonhomogeneous plasma. Since then, many papers have been devoted to developing rigorously the qualitative theory or asymptotic behavior for

$$\varrho(x)\partial_t u(x,t) = \Delta u^m(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.3}$$

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usually asking that $\varrho(x) \sim |x|^{-\gamma}$ as $|x| \to \infty$, for some $\gamma > 0$ (e.g. [5]), and even for the doubly nonlinear case

$$\varrho(x)\partial_t u(x,t) = \operatorname{div}\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right),$$

as in [6]. Thus, for $m > 1$, it has been noticed that while $0 < \gamma < 2$, the solutions have similar properties to the ones of the pure porous medium equation (for short, PME)

$$\partial_t u = \Delta u^m,$$

see [7,8], while for $\gamma > 2$, the properties of the solutions depart strongly from the ones of the PME [9]. Thus, $\gamma = 2$ is critical, and the asymptotic behavior for this case is left open in [9] with a conjecture giving the explicit profile. On the other hand, the asymptotic behavior for (1.3) with a density $\varrho(x) \sim |x|^{-2}$ at infinity, but ϱ regular near the origin, is studied in another recent work [10], obtaining an explicit profile solving (1.2), and proving the convergence towards it in the outer region (that is, outside small compacts near $x = 0$). Unusually, the linear diffusion problem $m = 1$ has been studied later than its nonlinear version, see for example [11,12].

In all cases, it has been noticed that the solutions converge asymptotically towards profiles coming from the pure power density equation, that is, $\varrho(x) = |x|^{-\gamma}$. Moreover,

$$|x|^{-\gamma}\partial_t u(x,t) = \Delta u^m(x,t) \tag{1.4}$$

has a more interesting feature: a singularity at $x = 0$, apart from the decay at infinity. Recently, in [13] the authors study formally some properties of radial solutions to (1.4) as a first step to understand its general behavior. Moreover, a study of existence and uniqueness for (1.4) for $\gamma > 2$ and $m > 1$ is done in [9, Section 6].

Coming back to our problem, that of letting $m = 1$, $\gamma = 2$ in (1.4), we find explicit asymptotic profiles which explain better the effect of the singularity at $x = 0$. The general case $m > 1$ will be treated in the companion paper [1]. But in order to explain these comments and to make precise the motivation for this work, let us state the main results of the paper.

Main results. We deal with the Cauchy problem associated with Eq. (1.1) with initial data

$$u_0 \in L^1_2(\mathbb{R}^N), \quad u_0 \geq 0, \tag{1.5}$$

where the dimension is $N \geq 3$ (except when specified) and

$$L^1_2(\mathbb{R}^N) := \left\{h : \mathbb{R}^N \mapsto \mathbb{R},\ h \text{ measurable}, \int_{\mathbb{R}^N} |x|^{-2}h(x)\,dx < \infty\right\}.$$

In Section 2 we give the precise notions of *weak solution* and *strong solution* to (1.1) and we prove that the Cauchy problem for (1.1) is well-posed in our framework. We refer the interested reader to Definition 2.1 and Theorem 2.2 for the precise statements.

We state the results about the large-time behavior, that we find quite interesting and unexpected. We begin with the case when $u_0(0) = 0$.

Theorem 1.1. *Let u be a* radially symmetric *solution of Eq.* (1.1) *with initial data satisfying* (1.5) *and moreover*

$$M_{u_0} := \int_{\mathbb{R}^N} |x|^{-N}u_0(x)\,dx < \infty, \qquad u_0(0) = 0. \tag{1.6}$$

Then we have

$$\lim_{t\to\infty} t^{1/2}\left\| u(x,t) - \frac{M_{u_0}}{\omega_1}F(x,t)\right\|_\infty = 0, \tag{1.7}$$

where ω_1 is the area of the unit sphere in $\mathbb{R}^N$ and

$$F(x,t) := \begin{cases} \dfrac{1}{\sqrt{4\pi t}}G\left(\dfrac{\log|x| + (N-2)t}{2\sqrt{t}}\right),\ G(\xi) = e^{-\xi^2}, & \text{for } |x| \neq 0, \\ 0, & \text{for } x = 0. \end{cases} \tag{1.8}$$

Remarks. (a) Let us notice also that

$$\max\{F(x,t) : x \neq 0\} = O(t^{-1/2}), \quad \text{as } t \to \infty,$$

showing that the time-scale $t^{1/2}$ is the correct one for the asymptotic behavior in (1.7). More precisely,

$$\|F(\cdot,t)\|_\infty = \frac{1}{\sqrt{4\pi t}}, \tag{1.9}$$

as will be analyzed in the remarks at the end of Section 4.

(b) The mass M_{u_0} in (1.6) is conserved along the flow. This will be obvious from the proof of Theorem 1.1.