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Sensitivity analysis for relaxed optimal control problems with final-state constraints

J. Frédéric Bonnans [a,b], Laurent Pfeiffer [a,b], Oana Silvia Serea [c,*]

[a] INRIA-Saclay, France

[b] CMAP, Ecole Polytechnique, 91128 Palaiseau Cedex, France

[c] Université de Perpignan Via Domitia, LAboratoire de Mathématiques et de PhySique, EA 4217, 52 avenue Paul Alduy, 66860 Perpignan Cedex, France

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ABSTRACT

In this article, we compute a second-order expansion of the value function of a family of relaxed optimal control problems with final-state constraints, parameterized by a perturbation variable. In this framework, relaxation with Young measures enables us to consider a wide class of perturbations and therefore to derive sharp estimates of the value function. The sensitivity analysis is performed in a neighborhood of a local optimal solution of a reference problem. The local solution $\bar{u}$ is assumed to be optimal with respect to the set of feasible relaxed controls having their support in a ball of a given radius $R > \|\bar{u}\|_\infty$ and having an associated trajectory very close to the reference trajectory, for the L^∞-norm. We call such a solution a relaxed R-strong solution.

1. Introduction

We consider a family of relaxed optimal control problems with final-state constraints, parameterized by a perturbation variable θ. The variable θ can perturb the dynamic of the system, the cost function and the final-state constraints. The aim of the article is to compute a second-order expansion of the value $V(\theta)$ of the perturbed problems, in the neighborhood of a reference value of θ, say $\bar{\theta}$. We assume that the reference problem has a classical local solution $\bar{u}$. The specificity of our work is to consider that this solution is an R-strong solution, a type of solutions that we introduce and which is closely related to the usual bounded strong solutions. We also provide some information on the first-order behavior of perturbed solutions.

There is already an important literature on sensitivity analysis of optimal control problems. Malanowski and Maurer prove the existence of weak solutions to the perturbed problems and their Fréchet-differentiability with respect to the perturbation parameter [1], by using a shooting formulation of the problems and extensions of the implicit function theorem. The previous work is concerned with optimal control problems with first-order state constraints and mixed constraints. The obtained derivative is itself the solution of a linear quadratic optimal control problem. A second-order expansion of the value

* Corresponding author. Tel.: +33 250505364.
E-mail addresses: frederic.bonnans@inria.fr (J. Frédéric Bonnans), laurent.pfeiffer@polytechnique.edu (L. Pfeiffer), oserea@yahoo.fr, oana-silvia.serea@univ-perp.fr (O.S. Serea).

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function is also obtained. We also refer the reader to [2] (and the references therein) for results on the Lipschitzian behavior of perturbed solutions of problems with mixed and first-order state constraints and to [3] for the case of problems with second-order (and higher-order) state constraints. Roughly speaking, three kinds of assumptions in all these papers are considered: a sufficient second-order condition, a qualification condition and strict complementarity conditions (imposing in particular the uniqueness of the multiplier).

In this article, rather than using the implicit function theorem, we follow the methodology described in [4] and originally in [5]. This approach allows to derive a second-order expansion of the value function without the assumptions of strict complementarity. In general, this method does not ensure differentiability properties of perturbed solutions. We proceed in the following manner: we begin by linearizing the family of optimization problems in the neighborhood of an optimal solution of the reference problem. Under a qualification condition, the first-order and second-order linearizations provide a second-order upper estimate of the value function. The two coefficients involved are the values of two linearized optimization problems, considered in their dual form. Then, a first lower estimate is obtained by expanding the Lagrangian up to the second order. Considering a strong sufficient second-order condition, we show that the distance between the reference solution and solutions to the perturbed problems is of order $|\theta - \bar{\theta}|$. Finally, the lower estimate corresponds to the upper estimate previously obtained.

The sensitivity analysis is performed in the framework of relaxed optimal controls. Roughly speaking, at each time, the control variable is not anymore a vector in a space U, but a probability measure on U, like if we were able to use several controls simultaneously. The new control variable is now a Young measure, in reference to the pioneering work of Young [6]. Relaxation of optimal control problems with Young measures has been much studied, in particular in [6–10]. Any Young measure is the weak-$*$ limit of a sequence of classical controls, therefore, we expect that a classical optimal control problem and its relaxed version have the same value. This question is studied, for instance, in [11,12].

Three aspects motivate the use of the relaxation. First, by considering convex combinations of controls in the sense of measures, we manage to describe in a convenient way a large class of tangential directions of the reachable set. This class of tangential directions was called *cone of variations* in the early papers of McShane [8], Gamkrelidze [13] and Warga [9,14]. It enables to prove Pontryagin's principle with the standard methods used to derive first-order optimality conditions of optimization problems. In our study, we obtain upper estimates expressed with Pontryagin multipliers. More precisely, the two linearized optimization problems that we obtain have a dual form involving multipliers for which Pontryagin's principle holds. Let us mention that Dmitruk used a partial relaxation technique in [15], under the name of *sliding modes*, to prove Pontryagin's principle. His method does not need the use of Young measures, since the relaxation is performed on discrete sets. On the other hand, an infinite sequence of auxiliary problems (justified in [16]) is required to obtain Pontryagin's multipliers. Second, in the framework of relaxation, we can derive a metric regularity theorem for the L^1-distance using abstract results from [17] and finally, the existence of relaxed solutions for the perturbed problem is guaranteed. Note that such solutions do not always exist in a classical framework.

The sensitivity analysis is realized locally, in a neighborhood of a local optimal solution $\bar{u}$ of the reference problem. In this study, we use the notion of relaxed R-strong optimal controls, for $R > \|\bar{u}\|_\infty$. We say that a control is a relaxed R-strong optimal solution if it is optimal with respect to the Young measures having their support in a ball of radius R and having a state variable sufficiently close for the uniform norm. This notion is related to the one of bounded strong solutions [18]. In order to obtain a sharp upper estimate of V, we must derive a linearized problem from a wide class of perturbations of the control. More precisely, we must be able to perturb the reference optimal control with close controls for the L^1-distance, taking into account that they are usually not necessarily close for the L^∞-distance. For such perturbations of the control, we use a particular linearization of the dynamics of the system, the Pontryagin linearization [18].

We obtain a lower estimate of the value function by assuming a sufficient second-order condition having the same nature as the one in [19]. We assume that a certain quadratic form is positive and that the Hamiltonian satisfies a quadratic growth condition. In order to expand the Lagrangian up to the second-order, we split the controls into two parts, one accounting for the small of the control in the L^∞-distance and the other one accounting for the large variations. We obtain an extension of the decomposition principle described in [19] and a lower estimate which corresponds to the upper estimate obtained previously.

The outline of the paper is as follows. In Section 2, we prove some preliminary results and in particular, a metric regularity theorem. Note that we will always suppose that the associated qualification condition holds. In Section 3, we obtain a first-order upper estimate of V and in Section 4 a second-order upper estimate, given in Theorem 4.11. In Section 5, we prove the decomposition principle (Theorem 5.2) and we obtain the lower estimate (Theorem 5.9). Two examples are discussed in Section 6. The theoretical material related to Young measures is recalled in Appendix A of the appendix, with precise references from [20–23]. In Appendix B, we justify the use of relaxation. Some technical proofs are presented in Appendix C for completeness.

2. Formulation of the problem and preliminary results

2.1. Setting

In this part, we define the family of optimal control problems that we want to study. We also introduce the notion of *relaxed R-strong* solutions.