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Regularity of flows and optimal control of shear-thinning fluids

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A B S T R A C T

We study optimal control problems of systems describing the flow of incompressible shear-thinning fluids. Taking advantage of regularity properties of the flows, we derive necessary optimality conditions under a restriction on the optimal control.

1. Introduction

This paper deals with optimal control problems associated with a viscous, incompressible fluid described by the following partial differential equations that generalize the Navier–Stokes system

$$\begin{cases} -\nabla \cdot (\tau(Dy)) + y \cdot \nabla y + \nabla \pi = u & \text{in } \Omega, \\ \nabla \cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where y is the velocity field, π is the pressure, τ is the extra stress tensor, $Dy = \frac{1}{2}\left(\nabla y + (\nabla y)^T\right)$ is the symmetric part of the velocity gradient ∇y, u is the given body force and $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) is a bounded domain with boundary Γ. We assume that $\tau : \mathbb{S} \longrightarrow \mathbb{S}$ is a classical power law stress tensor of the form

$$\tau_R(\eta) = 2\nu \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \eta \quad \text{or} \quad \tau_L(\eta) = 2\nu \left(1 + |\eta|\right)^{\alpha-2} \eta$$

where ν and α are positive constants. (Here $\mathbb{S}$ consists of all symmetric $n \times n$-matrices.) We recall that a fluid is called shear-thickening if $\alpha > 2$ and shear-thinning if $1 < \alpha < 2$. For the special case $\tau(\eta) = 2\nu\eta$ ($\alpha = 2$), we recover the Navier–Stokes equation with viscosity coefficient $\nu > 0$.

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The paper is concerned with the following optimal control problem

$$\text{Minimize } J(u,y) = \frac{1}{2}\int_\Omega |y-y_d|^2\, dx + \frac{\lambda}{2}\int_\Omega |u|^2\, dx$$
$$\text{Subject to } (u,y) \in U_{ad} \times W_0^{1,\alpha}(\Omega) \text{ satisfies (1.1) for some } \pi \in L^\alpha(\Omega)$$

where y_d is some desired velocity field, λ is a positive constant, the set of admissible controls U_{ad} is a nonempty convex closed subset of $L^q(\Omega)$ with $q > n$ and $\frac{3n}{n+2} \le \alpha < 2$. Although the analysis of several results can be more general, in order to simplify the presentation, we will assume that $U_{ad} \subset \{v \in L^q(\Omega) \mid \|v\|_q \le U\}$ for some $U > 0$. Throughout the paper, the optimal control problem corresponding to τ_R will be denoted by (P_R) and the optimal control problem corresponding to τ_L will be denoted by (P_L).

The partial differential equations describing the considered class of fluids were first proposed in [1–3] as a modification of the Navier–Stokes system, and were similarly suggested in [4]. Existence of weak solutions in $W_0^{1,\alpha}(\Omega)$ was proved by both authors using compactness arguments and the theory of monotone operators for $\alpha \ge \frac{3n}{n+2}$.

This basic regularity may prove insufficient for deriving the necessary optimality conditions for control problems governed by these equations, especially when considering shear-thinning fluids. As a consequence of the combined effect of the convective term and the nonlinear stress tensor, the lack of regularity of the state variable creates some difficulties in connection with the local Lipschitz continuity (and thus with the Gâteaux differentiability) in adequate functional spaces of the control-to-state mapping and with the natural setting for the associated linearized equation and the adjoint state equation. These issues were overcome in the case of shear-thickening fluids treated in [5,6] by using a suitable functional setting involving weighted Sobolev spaces. The optimality conditions are obtained in both two-dimensional and three-dimensional cases, without assuming any further regularity on the state and without restraining the set of admissible controls. The only constraint concerns the optimal control.

The case of shear-thinning fluids is more delicate and the techniques in [5,6] do not apply. In [7], these difficulties were handled by introducing a family of smooth approximate control problems falling into the case $\alpha = 2$ and whose solutions converge towards a solution of the original problem. The properties of the approximate control-to-state mapping were carefully studied and the approximate optimality conditions established. Under a constraint concerning the size of the optimal control, the same that guarantees uniqueness of the corresponding state, the optimality conditions for the original problem were then established by passing to the limit. As expected, because of the reduced regularity of the corresponding state variable, the adjoint equation is to be understood in the sense of distribution and uniqueness of the adjoint state is not guaranteed.

These issues can be more easily managed if the velocity gradient is bounded. Nevertheless, despite the fact that system (1.1) was widely studied, higher global regularity of solutions is difficult to obtain in general and there are only few such results known up to nowadays. In the case of steady shear-thinning fluids and C^1 extra stress tensors, the most significant global regularity results up to the boundary have been obtained in [8] in the two-dimensional framework enabling the derivation of some optimality conditions in [9], though restricting all the admissible controls to guarantee uniqueness of the corresponding solution and differentiability of the control-to-state mapping. The same considerations have led Wachsmuth and Roubíček [10], by exploiting the regularity results established in [11] for unsteady shear-thickening fluids, to restrict their studies to the case $n = 2$ and to restrain the set of admissible controls.

In the present work we follow [12], where both two-dimensional and three-dimensional cases for C^1 and Lipschitz continuous extra stress tensors were treated, and identify a condition under which uniqueness and regularity of weak solutions are both guaranteed. More precisely, we establish that if an admissible control u satisfies the following condition

$$\begin{cases} \text{There exists a positive constant } \kappa^* \text{ depending only on } n, \alpha, q \text{ and } \Omega \text{ such that} \\ \kappa^* \left(\frac{\|u\|_q}{\nu} + \left(1 + \frac{\|u\|_q}{\nu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|u\|_q}{\nu^2} \right) < 1 \end{cases} \tag{1.2}$$

then uniqueness and Hölder continuity of the corresponding state variable can be established, the corresponding adjoint state equation can be interpreted in the weak sense, the adjoint state being unique and more regular than stated in [7]. Assuming that the optimal control satisfies (1.2), well posed optimality conditions for problem (P_R) are then a direct consequence of the optimality conditions established in [7].

Concerning problem (P_L), besides the difficulties induced by the nonlinearity of the extra stress tensor and the convective term, the non-regularity of the model has to be managed. Since τ_L is not differentiable at the origin, the optimality conditions of [7] cannot be used. To overcome this difficulty, we introduce a family of regularized problems that fall into the case of C^1 extra stress tensors. Assuming that the optimal control satisfies (1.2) and exploiting some results of existence, uniqueness and regularity of solutions for the corresponding regularized state equation already established in [12], we derive the corresponding optimality systems and the optimality conditions for (P_L) are obtained by passing to the limit. A similar regularization approach was successfully used in [13] to study optimal control problems of systems governed by quasilinear elliptic equations with non differentiable coefficients at the origin and more recently in [14], for the treatment of a problem governed by the Bingham nonlinear mixed variational inequality.