Nonlinear Analysis 89 (2013) 110–120

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Lipschitz properties of nonsmooth functions and set-valued mappings via generalized differentiation

Nguyen Mau Nam *, Gerardo Lafferriere

Fariborz Maseeh Department of Mathematics and Statistics, Portland State University, PO Box 751, Portland, OR 97207, United States

ARTICLE INFO

Article history:
Received 20 February 2013
Accepted 1 May 2013
Communicated by S. Carl

MSC:
49J53
49J52
90C31

Keywords:
Lipschitz property
Generalized differentiation
Subdifferential
Coderivative
Minimal time function
Scalarization function

ABSTRACT

In this paper, we revisit the Mordukhovich subdifferential criterion for Lipschitz continuity of nonsmooth functions and the coderivative criterion for the Aubin/Lipschitz-like property of set-valued mappings in finite dimensions. The criteria are useful and beautiful results in modern variational analysis showing the state of the art of the field. As an application, we establish necessary and sufficient conditions for Lipschitz continuity of the minimal time function and the scalarization function, which play an important role in many aspects of nonsmooth analysis and optimization.

1. Introduction and preliminaries

Lipschitz continuity is an important concept in mathematical analysis. In modern variational analysis, it has been generalized for set-valued mappings. Among many extensions, the *pseudo-Lipschitzian property*, introduced by Aubin [1], has been well recognized as a natural and useful one. It is now called by different names, such as the *Aubin property* or the *Lipschitz-like property*. The concept has been used extensively in the study of sensitivity analysis of optimization problems and variational inequalities. It also plays an important role in developing generalized differentiation calculi for nonsmooth functions and set-valued mappings; see [2,3] and the references therein for more discussions on the history of the concept, as well as many important applications to variational analysis, optimization, and optimal control.

Recall that a set-valued mapping $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ has the *Aubin property* around $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F} := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid y \in \mathcal{F}(x)\}$ if there exist neighborhoods V of $\bar{x}$, W of $\bar{y}$, and a constant $\ell \geq 0$ such that

$$\mathcal{F}(x) \cap W \subseteq \mathcal{F}(u) + \ell\|x - u\|\mathbb{B} \quad \text{for all } x, u \in V.$$

The first effort to characterize the Aubin property using generalized differentiation was made by Rockafellar [4]. A sufficient condition for $\mathcal{F}$ to have the Aubin property around $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}$ was given as follows:

$$[(u, 0) \in N_C((\bar{x}, \bar{y}); \operatorname{gph} \mathcal{F})] \Rightarrow u = 0,$$

where $N_C((\bar{x}, \bar{y}); \operatorname{gph} \mathcal{F})$ is the *Clarke normal cone* to gph $\mathcal{F}$ at $(\bar{x}, \bar{y})$; see [5].

* Corresponding author. Tel.: +1 503 725 3659.
E-mail addresses: mau.nam.nguyen@pdx.edu (N.M. Nam), gerardoL@pdx.edu (G. Lafferriere).

0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.05.001

However, the Clarke normal cone is *too large* to be able to recognize the Aubin property of set-valued mappings in many different settings; see more discussions in [6,7]. Therefore, a natural question is to find a necessary and sufficient condition for this property using *smaller normal cone structures*, and the *Mordukhovich/limiting normal cone* [8] gives an answer to this question. The implication

$$[(u,0)\in N((\bar{x},\bar{y});\operatorname{gph}\mathcal{F})]\Rightarrow u=0 \tag{1.1}$$

in terms of the Mordukhovich normal cone $N((\bar{x},\bar{y});\operatorname{gph}\mathcal{F})$ to $\operatorname{gph}\mathcal{F}$ at $(\bar{x},\bar{y})$ is indeed a necessary and sufficient condition for $\mathcal{F}$ to have the Aubin property around $(\bar{x},\bar{y})$. This striking result was first proved by Mordukhovich in [6, Theorem 5.4]. It is now called the *Mordukhovich coderivative criterion* for the Aubin property. We will get back to the idea behind Mordukhovich's proof after presenting some important concepts of variational analysis. Readers are referred to [2,3] for more details.

Let $\mathcal{F}:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ be a set-valued mapping. The *domain* of the mapping is

$$\operatorname{dom}\mathcal{F}:=\{x\in\mathbb{R}^m\mid\mathcal{F}(x)\neq\emptyset\}.$$

Given $x\in\mathbb{R}^n$ and a subset $\Omega\subseteq\mathbb{R}^n$, the *distance function* from x to Ω is given by

$$d(x;\Omega):=\inf\{\|x-\omega\|\mid\omega\in\Omega\}.$$

The set

$$\Pi(x;\Omega):=\{\omega\in\Omega\mid d(x;\Omega)=\|x-\omega\|\}$$

is called the *metric projection* from x to Ω.

The following function defined on $\mathbb{R}^m\times\mathbb{R}^n$ will play an important role throughout the paper:

$$D(x,y):=d(y;\mathcal{F}(x)). \tag{1.2}$$

Let $\Omega\subseteq\mathbb{R}^n$, and let $\bar{x}\in\Omega$. A vector $v\in\mathbb{R}^n$ is called a *Fréchet normal* to Ω at $\bar{x}$ if

$$\limsup_{x\xrightarrow{\Omega}\bar{x}}\frac{\langle v,x-\bar{x}\rangle}{\|x-\bar{x}\|}\le 0.$$

The set of all Fréchet normals to Ω at $\bar{x}$ is called the *Fréchet normal cone* to Ω at $\bar{x}$, denoted by $\widehat{N}(\bar{x};\Omega)$.

A vector $v\in\mathbb{R}^n$ is called a *limiting normal* to Ω at $\bar{x}$ if there are sequences $x_k\xrightarrow{\Omega}\bar{x}$ and $v_k\to v$ with $v_k\in\widehat{N}(x_k;\Omega)$. In this definition, the notation $x_k\xrightarrow{\Omega}\bar{x}$ means that $x_k\to\bar{x}$ with $x_k\in\Omega$ for every k. The set of all limiting normals to Ω at $\bar{x}$ is called the *Mordukhovich/limiting normal cone* to the set at $\bar{x}$, and is denoted by $N(\bar{x};\Omega)$.

Let $\psi:\mathbb{R}^n\to(-\infty,\infty]$ be an extended real-valued function, and let $\bar{x}$ be an element of the *domain* of the function $\operatorname{dom}\psi:=\{x\in\mathbb{R}^n\mid\psi(x)<\infty\}$. The *Fréchet subdifferential* of ψ at $\bar{x}$ is defined by

$$\widehat{\partial}\psi(\bar{x}):=\left\{v\in\mathbb{R}^n\mid\liminf_{x\to\bar{x}}\frac{\psi(x)-\psi(\bar{x})-\langle v,x-\bar{x}\rangle}{\|x-\bar{x}\|}\ge 0\right\}.$$

The *limiting/Mordukhovich subdifferential* of ψ at $\bar{x}$, denoted by $\partial\psi(\bar{x})$, is the set of vectors $v\in\mathbb{R}^n$ such that there exist sequences $x_k\xrightarrow{\psi}\bar{x}$, and $v_k\in\widehat{\partial}\psi(x_k)$ with $v_k\to v$. The *singular subdifferential* of ψ at $\bar{x}$, denoted by $\partial^\infty\psi(\bar{x})$, is the set of all vectors $v\in\mathbb{R}^n$ such that there exist sequences $\lambda_k\downarrow 0$, $x_k\xrightarrow{\psi}\bar{x}$, and $v_k\in\widehat{\partial}\psi(x_k)$ with $\lambda_k v_k\to v$. In these definitions, $x_k\xrightarrow{\psi}\bar{x}$ means that $x_k\to\bar{x}$ with $\psi(x_k)\to\psi(\bar{x})$, and $\lambda_k\downarrow 0$ means that $\lambda_k\to 0$ with $\lambda_k\ge 0$ for every k. Both the Fréchet construction and the limiting subdifferential construction reduce to the subdifferential in the sense of convex analysis when the function involved is convex. Moreover, if ψ is lower semicontinuous around $\bar{x}$, one has the following representation:

$$\partial^\infty\psi(\bar{x})=\{v\in\mathbb{R}^n\mid(v,0)\in N((\bar{x},\psi(\bar{x}));\operatorname{epi}\psi)\}.$$

An extended real-valued function $\psi:\mathbb{R}^n\to(-\infty,\infty]$ is called *Lipschitz continuous* around $\bar{x}\in\operatorname{dom}\psi$ if there exist a constant ℓ and a neighborhood V of $\bar{x}$ such that

$$|\psi(x)-\psi(u)|\le\ell\|x-u\|\quad\text{for all } x,u\in V.$$

If this equality is replaced by

$$|\psi(x)-\psi(\bar{x})|\le\ell\|x-\bar{x}\|\quad\text{for all } x\in V,$$

we say that ψ is *calm* at $\bar{x}$.

Let $\mathcal{F}:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ be a set-valued mapping, and let $(\bar{x},\bar{y})\in\operatorname{gph}\mathcal{F}$. The *Mordukhovich/limiting coderivative* of $\mathcal{F}$ at $(\bar{x},\bar{y})$ is a set-valued mapping, denoted by $D^*\mathcal{F}(\bar{x},\bar{y}):\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ defined by

$$D^*\mathcal{F}(\bar{x},\bar{y})(v):=\{u\in\mathbb{R}^m\mid(u,-v)\in N((\bar{x},\bar{y});\operatorname{gph}\mathcal{F})\}.$$

The necessary and sufficient condition (1.1) for $\mathcal{F}$ to have the Aubin property around $(\bar{x},\bar{y})\in\operatorname{gph}\mathcal{F}$ can be equivalently represented in terms of the Mordukhovich coderivative in the theorem below.