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Exact boundary behavior of the unique positive solution to some singular elliptic problems☆

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In this paper, we give an exact asymptotic of the unique solution to the following singular boundary value problem $-\Delta u = a(x)g(u), x \in \Omega, u > 0$, in Ω, $u_{|\partial\Omega} = 0$. Here Ω is a C^2-bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $g \in C^1((0,\infty),(0,\infty))$ is nonincreasing on $(0,\infty)$ with $\lim_{t\to 0} g'(t)\int_0^t \frac{ds}{g(s)} = -C_g \leq 0$ and the function a is in $C^{\alpha}_{loc}(\Omega)$, $0 < \alpha < 1$ satisfying

$$0 < a_1 = \liminf_{d(x)\to 0} \frac{a(x)}{h(d(x))} \leq \limsup_{d(x)\to 0} \frac{a(x)}{h(d(x))} = a_2 < \infty,$$

where $h(t) = c\,t^{-\lambda}\exp(\int_t^{\eta} \frac{z(s)}{s} ds)$, $\lambda \leq 2$, $c > 0$ and z continuous on $[0,\eta]$ for some $\eta > 0$ such that $z(0) = 0$. Two applications of this result are also given. The first concerns the boundary behavior of the unique solution of $-\Delta u + \frac{\beta}{u}|\nabla u|^2 = a(x)g(u)$ that vanishes on the boundary and the second concerns the behavior of u in the case where the open set Ω is an annular and the behaviors of the function a on the interior boundary and the exterior boundary may be different.

1. Introduction

Let Ω be a C^2-bounded domain in $\mathbb{R}^n$, $n \geq 2$, a be a nonnegative function in $C^{\alpha}_{loc}(\Omega)$, $(0 < \alpha < 1)$ and g be a nonnegative nonincreasing function on $(0,\infty)$. The singular nonlinear Dirichlet problem

$$\begin{cases} -\Delta u = a(x)g(u), & x \in \Omega, \\ u > 0, & \text{in } \Omega, u_{|\partial\Omega} = 0, \end{cases} \tag{1.1}$$

has been extensively studied and the questions of existence, uniqueness and the boundary behavior are investigated. The problem (1.1) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials (see [1–4] and the references therein).

When $a = 1$ on Ω, Crandall, Rabinowitz and Tartar showed in [1] that problem (1.1) has a unique classical solution u in Ω such that

$$c_1\, p(d(x)) \leq u(x) \leq c_2\, p(d(x)), \quad \text{near the boundary of } \Omega,$$

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where c_1, c_2 are positive constants, $d(x)$ is the Euclidean distance from x to the boundary and p is the local nonnegative solution of the problem $-p''(s) = g(p(s))$ in $(0, \eta)$, $p(0) = 0$. In particular for $g(t) = t^{-\gamma}$, $\gamma > 1$, the solution u satisfies

$$c_1 \ (d(x))^{\frac{2}{1+\gamma}} \leq u(x) \leq c_2 \ (d(x))^{\frac{2}{1+\gamma}}, \quad \text{near the boundary of } \Omega. \tag{1.2}$$

In [5], Lazer and Mckenna showed that for $g(t) = t^{-\gamma}$ $(\gamma > 1)$ the inequality (1.2) continues to hold on $\overline{\Omega}$ and instead of $a = 1$ in Ω, they assume that

$$0 < b_1 \leq a(x)(d(x))^{\sigma} \leq b_2, \quad \text{for all } x \in \overline{\Omega},$$

where b_1, b_2 are positive constants and $\sigma \in (0, 2)$. Then they proved that for $\gamma > 1$, there exist positive constants c_1 and c_2 such that

$$c_1 \ (d(x))^{\frac{2}{1+\gamma}} \leq u(x) \leq c_2 \ (d(x))^{\frac{2-\sigma}{1+\gamma}}, \quad \text{for } x \in \overline{\Omega}.$$

When $a \in C^{\alpha}(\overline{\Omega})$ satisfies the following assumptions: there exist $\delta_0 > 0$ and a positive non-decreasing function $k_1 \in C(0, \delta_0)$ such that

(a_{01}) $\lim_{d(x)\to 0} \frac{a(x)}{k_1(d(x))} = a_0 \in (0, \infty)$,
(a_{02}) $\lim_{t\to 0^+} k_1(t)g(t) = \infty$;

and $g \in C^1((0, \infty), (0, \infty))$ with $\lim_{s\to 0^+} g(s) = \infty$ and g is non-decreasing on $(0, \infty)$ and satisfies the following conditions

(g_{01}) there exist positive c_0, η_0 and $\gamma \in (0, 1)$ such that $g(s) \leq c_0 s^{-\gamma}$, $\forall s \in (0, \eta_0)$;
(g_{02}) there exist $\theta > 0$ and $s_0 \geq 1$ such that $g(\xi s) \geq \xi^{-\theta} g(s)$ for all $\xi \in (0, 1)$ and $0 < s \leq s_0\xi$;
(g_{03}) the mapping $\xi \in (0, \infty) \to T(\xi) = \lim_{s\to 0^+} \frac{g(\xi s)}{\xi g(s)}$ is a continuous function;

Ghergu and Radulescu [6] showed that the unique solution u of problem (1.1) satisfies $u \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ and

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi_3(d(x))} = \xi_0, \tag{1.3}$$

where $T(\xi_0) = b_0^{-1}$ and $\psi_3 \in C^1([0, \delta_1]) \cap C^2((0, \delta_1])$ $(\delta_1 \in (0, \delta_0))$ is the local solution of the problem $-\psi_3''(t) = k_1(t)g(\psi_3(t))$, $\psi_3(t) > 0$, $0 < t < \delta_1$, $\psi_3(0) = 0$.

In [7], Zhang extended the result of Ghergu and Radulescu [6] to the case where g is normalized regularly varying at zero with index $-\gamma$ $(\gamma > 0)$ and k_1 in (a_{01}) is normalized regularly varying at zero with index $-\beta$ $(\beta \in (0, 2))$.

Recently, Ben Othman et al. [8] and Gontara et al. [9] extended the results of [6,7] to a large class of functions a which belongs to the Kato class $K(\Omega)$ and g is normalized regularly varying at zero with index $-\gamma$ $(\gamma \geq 0)$. In particular, they established an exact boundary behavior of the unique classical solution of the problem $-\Delta\omega = a(x)$, $\omega > 0$ in Ω, $\omega_{|\partial\Omega} = 0$, when a satisfies

(a_{03}) $a \in C^{\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$, is positive in Ω and

$$0 < \widetilde{a}_1 = \liminf_{d(x)\to 0} \frac{a(x)}{k_1(d(x))} \leq \limsup_{d(x)\to 0} \frac{a(x)}{k_1(d(x))} = \widetilde{a}_2 < \infty,$$

with

$$k_1(t) = t^{-2} \prod_{i=1}^{m} \left(\log_i(t^{-1})\right)^{-\mu_i}, \quad t \in (0, \delta_0), \tag{1.4}$$

where $\log_i(t^{-1}) = \log \circ \log \circ \log \circ \cdots \circ \log(t^{-1})$ and $\mu_1 = \mu_2 = \cdots = \mu_{j-1} = 1$, $\mu_j > 1$ and $\mu_i \in \mathbb{R}$ for $j + 1 \leq i \leq m$.

Inspired by some ideas in [8,9], the authors in [10] extend the previous results on the boundary behavior of the solution u of problem 1.1 to the case where k_1 is given by (1.4) or k_1 lies into a class of functions Λ that was introduced by Cîrstea and Rădulescu in [11] for non-decreasing functions and by Mohammed in [12] for nonincreasing functions as the set of positive monotonic functions $k \in C^1((0, \delta_0)) \cap L^1(0, \delta_0)$ $(\delta > 0)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = \rho_k \in [0, \infty), \quad K(t) = \int_0^t k(s)\, ds$$

and the function $g \in C^1((0, \infty), (0, \infty))$, decreasing on $(0, \infty)$ with $\lim_{s\to 0^+} g(s) = \infty$ and satisfies

(g_{04}) there exists $C_g > 0$ such that $\lim_{s\to 0^+} g'(s) \int_0^s \frac{d\nu}{g(\nu)} = -C_g$.

They proved separately, under these conditions on g, the following two theorems.