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Existence and non-existence of positive solutions for nonlinear elliptic singular equations with natural growth

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ABSTRACT

In this paper we analyze the existence, non-existence and uniqueness of positive solutions of some nonlinear elliptic equations containing singular terms and natural growth in the gradient term. We use an adequate sub–super-solution method to prove the existence of solutions, the characterization of eigenvalues and the integrability of the term $|\nabla u|^2/u^2$ for the non-existence and results from Arcoya–Segura de León for the uniqueness.

1. Introduction

We study existence, non-existence and uniqueness of positive solutions to the following nonlinear elliptic problem

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = f(\lambda, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 3$) and λ is a real parameter. The functions $f \in C(\mathbb{R} \times [0, +\infty))$ and $g \in C(0, +\infty)$ are given, for some $k, \gamma, p, q > 0$, by

$$g(s) = \frac{k}{s^\gamma}, \quad \forall s > 0, \tag{1.2}$$

$$f(\lambda, s) = \lambda s^q \quad \text{or} \quad f(\lambda, s) = \lambda s - s^p, \quad \forall s > 0. \tag{1.3}$$

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We say that a solution of (1.1) is a function $u \in H_0^1(\Omega)$ such that $0 < u$ almost everywhere in $\Omega, g(u)|\nabla u|^2 \in L^1(\Omega), f(\lambda, u) \in L^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega g(u)|\nabla u|^2 \phi = \int_\Omega f(\lambda, u)\phi$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

This kind of equations (with quadratic gradient terms) has attracted much interest since the pioneering works [1,2]. In the last years, attention has been paid in singular terms in front of the gradient terms [3–8]. In fact, in most of these papers the source term is not identically zero, i.e. for different kind of nonnegative functions g and f, is studied the equation

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = f(\lambda, u) + f_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $0 \le f_0 \in L^r(\Omega)$ $(r \ge 1)$ and $f_0 \not\equiv 0$. Thanks to this fact some aspects in the study of existence of solution of (1.4) can be simplified since the singularity can be avoided in compactly embedded subsets of Ω.

When $f(\lambda, u) \equiv 0$ and g is continuous at zero, problem (1.4) was studied, among others, in [1,2]. This was the starting point in [5,7,8] for the singular case, to consider $f(\lambda, u) \equiv 0$ and $g(s)$ singular at zero, as the model k/s^γ. In [3] the authors showed the existence of positive solutions for $\gamma < 2$ and non-existence for $\gamma \ge 2$. Moreover, in [6] it is proved that the solution is unique in the case $\gamma < 1$.

When $f(\lambda, s) = \lambda s$ and $g \equiv 1$, in [9] it was proved that there exists a positive solution of (1.4) for every $\lambda > 0$ showing the regularizing effect of the quadratic gradient term. In [4] it was proved that this regularizing effect remains true while

$$g(s) \ge k/s^\gamma, \quad \gamma < 1, \text{ for } s \text{ large enough.} \tag{1.5}$$

In particular, for $f(\lambda, s) = \lambda s, g(s) = \frac{k}{s^\gamma}$ with $\gamma < 1$ and $f_0 \in L^{\frac{2N}{N+2}}(\Omega)$, in [4], using bifurcation theory, the existence of a positive solution of (1.4) for every $\lambda > 0$ was proved. In [10], by means of an approximative scheme, the existence of a solution for more general divergence operators and less regular data $f_0 \in L^m(\Omega)$ with $m \ge \frac{2N}{2N-\gamma(N-2)}$ was proved.

For $f(\lambda, s) = \lambda s^q$ and $g(s) = k/s^\gamma$ with $0 < \gamma < 1$ and $\gamma + q < 2$ the existence of a positive solution of (1.4) for every $\lambda > 0$ was proved in [10], see also [4].

Very little is known for the problem (1.4) if $f_0 \equiv 0$, mostly in the case of functions g that are continuous at zero. In this case [4] the existence of a positive solution for every $\lambda > 0$ if $f(\lambda, s) = \lambda s^q$ with $q < 1$ is proved. If $q = 1$ and g is a continuous function satisfying (1.5), in [4] the existence of a solution for $\lambda > \lambda_1$, where λ_1 denotes the first eigenvalue of $-\Delta$ under homogeneous Dirichlet boundary conditions is obtained. In [11], for $f(\lambda, s) = \lambda s^q$ with $q > 1$ and given a continuous nonnegative function g satisfying (1.5), the existence of a positive solution if and only if $\lambda \ge \lambda^*$ for some $\lambda^* > 0$ is proved. Moreover, the multiplicity of solutions is shown for $\lambda > \lambda^*$. Finally if $g \equiv 1$, the case $f(\lambda, s) = \lambda s - s^p$, for some $p > 1$, was analyzed in [12] showing the existence and uniqueness of continuously differentiable positive solution for $\lambda > \lambda_1$.

In this paper we study problem (1.4) with $f_0 \equiv 0$ in the case of functions g that are singular at zero. More precisely, for the sake of clarity, we consider problem (1.1) with functions g and f given respectively by (1.2) and (1.3). We summarize the main results here. We would like to point out that the first result contains direct results but we include it for the reader's convenience.

Theorem 1.1. *Assume that $f(\lambda, s) = \lambda s$ and $g(s) = \frac{k}{s^\gamma}$.*

1. *If $\gamma < 1$, there exists a positive solution of (1.1) if and only if $\lambda > \lambda_1$. Moreover, for $\lambda > \lambda_1$ there exists a unique bounded positive solution.*
2. *If $\gamma = 1$ and $k < 1$, then there exists a positive solution of (1.1) if and only if $\lambda = \lambda_1/(1-k)$. In this case, there exist infinite positive solutions.*
3. *If $\gamma = 1$ and $k \ge 1$, then (1.1) has no positive solution for $\lambda > 0$.*
4. *If $\gamma > 1$ then (1.1) has no positive solution for $\lambda > 0$.*

Theorem 1.2. *Assume that $f(\lambda, s) = \lambda s^q$, $0 < q < 1$ and $g(s) = \frac{k}{s^\gamma}$.*

1. *If $\gamma < 1$, there exists a unique bounded positive solution of (1.1) for every $\lambda > 0$.*
2. *If $\gamma = 1$ and $k \le q$, there exists at most one positive solution for every $\lambda > 0$.*
3. *If $\gamma + q > 2$, then (1.1) has no positive solution for $\lambda > 0$.*

Theorem 1.3. *Assume that $f(\lambda, s) = \lambda s^q$, $q > 1$ and $g(s) = \frac{k}{s^\gamma}$.*

1. *If $\gamma < 1$ and $\gamma + q < 2$, there exists $\lambda^* > 0$ such that (1.1) possesses a positive solution for every $\lambda \ge \lambda^*$ and (1.1) does not possess any positive solution for every $\lambda < \lambda^*$.*
2. *If $\gamma \ge 1$ then (1.1) has no positive solution for $\lambda > 0$.*