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The stability for a one-dimensional wave equation with nonlinear uncertainty on the boundary[☆,☆☆]

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ABSTRACT

In this work, we are concerned with the boundary stabilization of a one-dimensional wave equation subject to boundary nonlinear uncertainty. The nonlinear uncertainty is first estimated in terms of the output, and then canceled by its estimates. We show that this strategy works well when the derivative of the uncertainty is bounded.

1. Introduction

In this work, we are concerned with the stabilization of the following one-dimensional wave equation:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), & x \in (0,1), t > 0, \\ u(0,t) = 0, & t \ge 0, \\ u_x(1,t) = U(t) + d(t), & t \ge 0, \\ Y(t) = \displaystyle\int_0^1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u(x,t)dx, \end{cases} \tag{1.1}$$

where $u(x,t)$ is the state, $U(t)$ is the control input, and $Y(t)$ is the output. The nonlinear uncertainty $d(t)$ is supposed to be bounded measurable, that is, $|d(t)| \le M_0$ for some $M_0 > 0$ and all $t \ge 0$. When the uncertainty $d(t)$ is absent, it is easy to obtain that system (1.1) can be stabilized by choosing $U(t) = -u_t(1,t)$ [1]. However, since the nonlinear uncertainty is almost everywhere in the real world systems, it is very necessary to establish the stability of the system in the presence of uncertainty. In [2], Guo considered the stability of the following one-dimensional anti-stable wave equation in the presence of disturbance:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), & x \in (0,1), t > 0, \\ u_x(0,t) = -qu_t(0,t), & t \ge 0, \\ u_x(1,t) = U(t) + d(t), & t \ge 0, \end{cases} \tag{1.2}$$

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where $u(x,t)$ is the state, $U(t)$ is the control input, $0 < q \neq 1$ is a constant number, and $d(t)$ is the unknown disturbance. The main idea of Guo's strategy is that the disturbance is first estimated in terms of the output, and then canceled by its estimates. This control strategy is also called the active disturbance rejection control strategy, which has been used in [3–7].

It is seen from [4,5,7] that one of the main difficulties of the active disturbance rejection control strategy is the well-posedness of the closed-loop system. There are some classical works in connection with this subject. We refer the reader to [8–17].

Motivated mainly by [2], we apply Guo's method to system (1.1). The objective of our work is to design a continuous controller $U(t)$, which is based on the output $Y(t)$, to stabilize system (1.1) in the presence of disturbance.

We will consider systems (1.1) in the state space $\mathscr{H} = H_L^1(0,1) \times L^2(0,1)$, where $H_L^1(0,1) := \{v \in H^1(0,1) \mid v(0) = 0\}$. The energy of the system is defined by

$$E(t) = \frac{1}{2}\int_0^1 |u_t(x,t)|^2 dx + \frac{1}{2}\int_0^1 |u_x(x,t)|^2 dx. \tag{1.3}$$

Throughout this paper, we define

$$\|v(t)\|_\infty := \sup_{t\in[0,\infty)} |v(t)|. \tag{1.4}$$

Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. Define

$$\|X\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{1.5}$$

We denote by $\|A\|_F$ the Frobenius norm of the matrix A.

2. Main result and proof

Thanks to our choice of the output $Y(t)$, we have

$$\int_0^1 \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u_x(x,t)dx = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u(x,t)\Big|_0^1 - \frac{\pi}{2}\int_0^1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u(x,t)dx = -\frac{\pi}{2}Y(t) \tag{2.1}$$

and

$$\begin{aligned}\int_0^1 \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u_x(x,t)dx &= -\frac{2}{\pi}\cos\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u_x(x,t)\Big|_0^1 + \frac{2}{\pi}\int_0^1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u_{xx}(x,t)dx \\ &= \frac{2}{\pi}u_x(1,t) + \frac{2}{\pi}\int_0^1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) u_{tt}(x,t)dx \\ &= \frac{2}{\pi}[d(t) + U(t)] + \frac{2}{\pi}\ddot{Y}(t).\end{aligned} \tag{2.2}$$

Consequently,

$$\ddot{Y}(t) + \frac{\pi^2}{4}Y(t) + d(t) + U(t) = 0. \tag{2.3}$$

It is seen that (2.3) is an ODE with state Y and control U. Then we are able to design a state observer to estimate Y, $\dot{Y}$ and $-d$ as follows [18]:

$$\begin{cases}\dot{\widehat{Y}}_1(t) = \widehat{Y}_2(t) + \dfrac{3}{\varepsilon}(Y(t) - \widehat{Y}_1(t)), \\ \dot{\widehat{Y}}_2(t) = \widehat{Y}_3(t) - \dfrac{\pi^2}{4}\widehat{Y}_1(t) + \left(\dfrac{6}{\varepsilon^2} - \dfrac{\pi^2}{4}\right)(Y(t) - \widehat{Y}_1(t)) - U(t), \\ \dot{\widehat{Y}}_3(t) = \dfrac{6}{\varepsilon^3}(Y(t) - \widehat{Y}_1(t)),\end{cases} \tag{2.4}$$

where ε is the tuning small parameter. Combining (2.3) and (2.4), the errors

$$\widetilde{Y}_1(t) := \widehat{Y}_1(t) - Y(t), \qquad \widetilde{Y}_2(t) := \widehat{Y}_2(t) - \dot{Y}(t), \qquad \widetilde{Y}_3(t) := \widehat{Y}_3(t) + d(t) \tag{2.5}$$

satisfy

$$\begin{cases}\dot{\widetilde{Y}}_1(t) = \widetilde{Y}_2(t) - \dfrac{3}{\varepsilon}\widetilde{Y}_1(t), \\ \dot{\widetilde{Y}}_2(t) = \widetilde{Y}_3(t) - \dfrac{6}{\varepsilon^2}\widetilde{Y}_1(t), \\ \dot{\widetilde{Y}}_3(t) = -\dfrac{6}{\varepsilon^3}\widetilde{Y}_1(t) + \dot{d}(t).\end{cases} \tag{2.6}$$