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Homoclinic solutions of a class of periodic difference equations with asymptotically linear nonlinearities[☆]

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ABSTRACT

By using critical point theory and periodic approximations, new sufficient conditions are obtained on the existence and nonexistence of homoclinic solutions for a class of discrete nonlinear periodic equations with asymptotically linear nonlinearities. These results partially answer an open problem proposed by Pankov (2006) [2] under rather weaker conditions and greatly improve the related results before.

1. Introduction and main results

The aim of this paper is to study the existence of homoclinic solutions for a class of periodic difference equations

$$Lu_n - \omega u_n = \sigma g_n(u_n), \quad n \in Z, \tag{1.1}$$

where $\sigma = \pm 1$, $g_n(u)$ is continuous in u and with saturable nonlinearity, $g_{n+T}(u) = g_n(u)$ for all $n \in Z$, T is a positive integer, and L is the Jacobi operator given by

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n, \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are real valued T-periodic sequences.

The operator L is a bounded and self-adjoint operator in l^2. Its spectrum $\sigma(L)$ has a band structure, that is, $\sigma(L)$ is a union of a finite number of closed intervals (see, e.g. [1]). Thus the complement $R\backslash\sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite which are denoted by $(-\infty, \beta)$ and $(\alpha, +\infty)$, respectively.

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As usual, a solution $u = \{u_n\}$ of (1.1) is said to be homoclinic (to 0) if

$$\lim_{|n|\to+\infty} u_n = 0.$$

In addition, if $u_n \equiv 0$, then u is called the trivial solution, otherwise u is called a nontrivial homoclinic solution. Obviously, the nontrivial solution in l^2 of (1.1) is the nontrivial homoclinic solution of (1.1).

We prefer to study the existence of the nontrivial solutions in l^2 of (1.1). This problem originates with Pankov [2,3] by studying the gap solitons of the periodic discrete nonlinear Schrödinger equation

$$i\dot{\psi}_n = -\Delta\psi_n + \epsilon_n\psi_n - \sigma g_n(\psi_n), \quad n \in Z, \tag{1.3}$$

where $\sigma = \pm 1$, $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, the given sequences $\{\epsilon_n\}$ and g_n are assumed to be T-periodic. Since solitons are spatially localized time-periodic solutions and decay to zero at infinity, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t}$$

and

$$\lim_{|n|\to+\infty} u_n = 0.$$

Thus the problem on the existence of solitons of (1.3) has been reduced to the existence of solutions of the following equation

$$-\Delta u_n + \epsilon_n u_n - \omega u_n = \sigma g_n(u_n)$$

in l^2.

The discrete nonlinear Schrödinger (DNLS) equation is a nonlinear lattice system that occurs in many areas of physics such as nonlinear optics, biomolecular chains and Bose–Einstein condensates; see [4–6]. In the past decade, the existence of discrete solitons of the DNLS equations has become a hot topic, to mention a few, see [7–12,2,3,13–17]. Among the methods used are variational methods, center manifold reduction and Nehari manifold approach and so on. Many of these papers considered the DNLS equations with constant coefficients, and their results have been summarized in [18–20]. Recently, the DNLS equations with periodic coefficients have appeared in physics literature, and such phenomenon can be found by numerical simulation [21].

In 2006, Pankov [2] considered a special case of (1.3) with $g_n(u) = \chi_n u^3$. In the end of [2], Pankov posed an open problem on the existence of gap solitons for the asymptotically linear nonlinearities such as

$$g_n(u) = \chi_n u^3(1 + c_n u^2)^{-1}$$

and

$$g_n(u) = \chi_n(1 - \exp(-a_n u^2))u.$$

For these two special cases, Zhou et al. [16,17] and Shi et al. [13,14] provided some sufficient conditions on the existence of the gap solitons or homoclinic solutions of Eq. (1.1), and partially solved the open problem. Especially, when ω belongs to the semi-infinite spectral gap of L, Zhou et al. [15] studied the existence of homoclinic solutions of (1.1) by the Mountain Pass Lemma and periodic approximations. However, the most interesting case is that the frequency ω belongs to a finite gap, which has not been solved yet. Therefore, this paper mainly deals with the existence of nontrivial homoclinic solutions of (1.1) in l^2 in the case where ω belongs to finite or semi-infinite spectral gap (α, β) of L.

Throughout this paper, we always assume that $g_n(u)$ satisfies the following conditions.

(f_1) g_n is continuous in u, and $g_{n+T}(u) = g_n(u)$ for any $n \in Z$ and $u \in R$.
(f_2) $g_n(u)u - 2G_n(u) \to +\infty$ as $|u| \to +\infty$, where $G_n(u) = \int_0^u g_n(s)ds$ for $u \in R$.
(f_3) $g_n(u)/u$ is strictly increasing in $(0, +\infty)$ and $g_n(u)/u$ is strictly decreasing in $(-\infty, 0)$.

Moreover,

$$\lim_{|u|\to 0} \frac{g_n(u)}{u} = 0 \quad \text{and} \quad \lim_{|u|\to+\infty} \frac{g_n(u)}{u} = d_n < +\infty. \tag{1.4}$$

By periodicity of $g_n(u)$ in n and definition of d_n for $n \in Z$, the maximum and minimum of d_n can be achieved. Let d^* and d_* denote them respectively.

It is easily checked that $g_n(u) = \chi_n u^3(1+c_n u^2)^{-1}$ with positive and periodic coefficients satisfies the conditions (f_1), (f_2) and (f_3).

Now we are in a position to state our main results.

Theorem 1.1. *Assume that $\sigma = 1$, $\beta \neq +\infty$ and $\omega \in (\alpha, \beta)$, where (α, β) is a spectral gap of L. If $g_n(u)$ satisfies (f_1)–(f_3) with $d_* > \beta - \omega$, then (1.1) has at least a nontrivial solution u in l^2. Moreover, the solution decays exponentially at infinity, that is, there exist two positive constants C and τ such that*

$$|u_n| \le Ce^{-\tau|n|} \quad \textit{for } n \in Z. \tag{1.5}$$