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Monotonicity formulas via the Bakry–Émery curvature

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ABSTRACT

We prove several monotone formulas for smooth metric measure spaces with nonnegative m-Bakry–Émery curvature. First, we define $A_f(r)$ and $V_f(r)$ for the weighted Green function, and establish the monotone formulas. These formulas are parallel to the weighted volume comparison theorem on smooth metric measure spaces with nonnegative m-Bakry–Émery curvature. Second, we define the weighted Nash entropy, Fisher information and Perelman's entropy. Then we get the monotone formulas along the weighted heat equation via the m-Bakry–Émery curvature.

1. Introduction

Let M be an n-dimensional complete Riemannian manifold with metric g and $\mathrm{d}\mu(x) = e^{-f(x)}\,\mathrm{d}x$ the weighted measure, where f is a smooth potential function and dx is the Riemann–Lebesgue measure. The m-Bakry–Émery curvature is defined by

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m-n}\mathrm{d}f \otimes \mathrm{d}f,$$

where $m \geq n$ and $m = n$ only when f is a constant. The m-Bakry–Émery curvature is always used to replace the Ricci curvature when studying the weighted Laplacian [1–7].

We use $\triangle$ to denote the classical Laplacian determined by g. Then the weighted Laplacian is defined by

$$\triangle_f = \triangle - \nabla f \cdot \nabla.$$

It is easy to see that $\triangle_f$ is symmetric with respect to the weighted measure dμ. In fact,

$$\int_M \nabla u \cdot \nabla v \,\mathrm{d}\mu = -\int_M u \triangle_f v \,\mathrm{d}\mu$$

holds for any $u, v \in C_0^\infty(M)$. $\triangle_f$ relates to the m-Bakry–Émery curvature via the following weighted Bochner formula [1–3,7]

$$\frac{1}{2}\triangle_f |\nabla u|^2 = |\nabla^2 u|^2 + \nabla u \cdot \nabla \triangle_f u + \mathrm{Ric}_{f,m}(\nabla u, \nabla u) + \frac{1}{m-n}(\nabla f \cdot \nabla u)^2. \tag{1}$$

The classical Bishop–Gromov's volume comparison theorem states that on a complete manifold with nonnegative Ricci curvature, $r^{-n}V(B_x(r))$ is monotone nonincreasing in the radius r for any fixed $x \in M$, where $V(B_x(r))$ denotes the volume of the geodesic ball centered at x with radius r. This theorem follows from integrating the Laplacian of the distance squared

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to a point [8]. As a parallel theory of Bishop–Gromov's volume comparison theorem, the monotone formulas for $A(r)$ and $V(r)$ were obtained by Colding in [9], where

$$A(r) = r^{1-n}\int_{b=r}|\nabla b|^3,$$
$$V(r) = r^{-n}\int_{b\le r}|\nabla b|^4,$$

where the integral is under the measure dx, and $G = b^{2-n}$ is the Green function with respect to the Laplacian $\triangle$.

Note that the classical Bishop–Gromov's volume comparison theorem has been generalized to the weighted measure case via the m-Bakry–Émery curvature [2,10,7]. The generalized one states that on a complete manifold with nonnegative m-Bakry–Émery curvature, $r^{-m}\mu(B_x(r))$ is monotone nonincreasing in the radius r for any fixed $x \in M$, where $\mu(B_x(r))$ denotes the dμ measure of $B_x(r)$. Hence we can establish a parallel theory of the weighted volume comparison theorem. We shall define $A_f(r)$ and $V_f(r)$, and prove several monotone formulas. We do these in Sections 2–4.

In [11], Perelman introduced the following W-functional

$$W(\mathrm{g}, t) = \int_M [t(\mathrm{R} + \triangle v) + v - n](4\pi t)^{-\frac{n}{2}}e^{-v}\,\mathrm{d}x,$$

where v satisfies

$$\int_M (4\pi t)^{-\frac{n}{2}}e^{-v}\,\mathrm{d}x = 1. \tag{2}$$

He also proved a monotone formula for $W(\mathrm{g}, t)$ along the Ricci flow. Then he could rule out the nontrivial shrinking breathers of the Ricci flow. Later Ni [12,13] defined

$$W(t) = \int_M (t|\nabla v|^2 + v - n)\frac{e^{-v}}{(4\pi t)^{\frac{n}{2}}}\,\mathrm{d}x,$$

where v satisfies (2). The main result in [12] states that $W(t)$ is monotone nonincreasing along the heat equation

$$\left(\frac{\partial}{\partial t} - \triangle\right)u = 0 \tag{3}$$

when the Ricci curvature is nonnegative. Based on this monotone result, Ni got a differential Harnack inequality for the heat kernel. Ni also pointed out in [13] that

$$W(t) = F(t) + N(t),$$

where

$$F(t) = t\int_M \frac{|\nabla u|^2}{u}\,\mathrm{d}x - \frac{n}{2}$$

and $N(t)$, the Nash entropy, is defined by

$$N(t) = -\int_M \log u u\,\mathrm{d}x - \frac{n}{2}\log(4\pi e t).$$

Note that $F(t)$ is closely related to the Li–Yau gradient estimate for heat equation (3) given in [14]. This estimate states that on a manifold with nonnegative Ricci curvature, a positive solution to (3) satisfies

$$|\nabla \log u|^2 - \partial_t \log u \le \frac{n}{2t}.$$

Integrating this inequality yields that $F(t) \le 0$. It is not hard to see that $\frac{F}{t}$ is the derivative of the Nash entropy $N(t)$ [13]. We can also find properties of $F(t)$ and $N(t)$ in [9].

In the second part of this paper, we define the weighted F_f, N_f and W_f functionals with respect to the weighted measure $\mathrm{d}\mu = e^{-f(x)}\mathrm{d}x$. We shall point out that $F_f(t)$ is closely related to the Li–Yau gradient estimate for the weighted heat equation

$$(\partial_t - \triangle_f)u = 0 \tag{4}$$

on manifolds with nonnegative m-Bakry–Émery curvature. This estimate was proved by Li in [1]. We also give several monotone properties for F_f, N_f and W_f. We do these in Section 5.