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Global bifurcation diagrams and exact multiplicity of positive solutions for a one-dimensional prescribed mean curvature problem arising in MEMS[☆]

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ABSTRACT

We study global bifurcation diagrams and exact multiplicity of positive solutions for the one-dimensional prescribed mean curvature problem arising in MEMS

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^p}, & u<1,\ -L<x<L, \\ u(-L)=u(L)=0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter, and $p, L > 0$ are two evolution parameters. We determine the exact number of positive solutions by the values of p, L and λ. Moreover, for $p \geq 1$, the bifurcation diagram undergoes fold and splitting bifurcations. While for $0 < p < 1$, the bifurcation diagram undergoes fold, splitting and segment-shrinking bifurcations. Our results extend and improve those of Brubaker and Pelesko [N.D. Brubaker, J.A. Pelesko, Analysis of a one-dimensional prescribed mean curvature equation with singular nonlinearity, Nonlinear Anal. 75 (2012) 5086–5102] and Pan and Xing [H. Pan, R. Xing, Exact multiplicity results for a one-dimensional prescribed mean curvature problem related to a MEMS model, Nonlinear Anal. RWA 13 (2012) 2432–2445] by generalizing the nonlinearity $(1-u)^{-2}$ to $(1-u)^{-p}$ with general $p \in (1, \infty)$. We also answer an open question raised by Brubaker and Pelesko on the extension of (global) bifurcation diagram results to general $p > 0$. Concerning this open question, we find and prove that global bifurcation diagrams for $0 < p < 1$ are different to and more complicated than those for $p \geq 1$.

1. Introduction

In this paper we study global bifurcation diagrams and exact multiplicity of positive solutions $u \in C^2(-L, L) \cap C[-L, L]$ for the one-dimensional prescribed mean curvature problem arising in electrostatic MEMS

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^p}, & u<1,\ -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.1}$$

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where $\lambda > 0$ is a *bifurcation* parameter, and $p, L > 0$ are two *evolution* parameters. The singular nonlinearity

$$f(u) \equiv \frac{1}{(1-u)^p}, \quad p > 0,$$

satisfies

$$f(0) = 1, \qquad \lim_{u \to 1^-} f(u) = \infty, \quad \text{and} \quad f'(u), f''(u) > 0 \text{ on } [0, 1). \tag{1.2}$$

Notice that the improper integral of f over $[0, 1)$ satisfies

$$\int_0^1 f(u)du = \begin{cases} \infty & \text{if } p \geq 1, \\ \dfrac{1}{1-p} < \infty & \text{if } 0 < p < 1. \end{cases}$$

The prescribed mean curvature problem

$$\begin{cases} -\text{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \lambda \tilde{f}(u), & u < 1, \mathbf{x} \in \Omega_L, \\ u = 0, \quad \mathbf{x} \in \partial \Omega_L, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a bifurcation parameter and $\Omega_L \subset \mathbb{R}^n (n \geq 1)$ is a smooth bounded domain depending on some parameter $L > 0$, with general nonlinearity $\tilde{f}(u)$ or with many different types nonlinearities, like $\exp(u)$, $(1+u)^p$ $(p > 0)$, $\exp(u) - 1$, u^p $(p > 0)$, a^u $(a > 0)$, $u - u^3$, and $u^p + u^q$ $(0 \leq p < q < \infty)$ has been recently investigated by many authors; see e.g. [1–16]. We note that, in particular, Bonheure et al. [2] is the first paper when the problem with singularities has been considered. Also note that, in geometry, a solution $u(x)$ of (1.3) is also called a graph of prescribed (mean) curvature $\lambda \tilde{f}(u)$.

A solution $u \in C^2(-L, L) \cap C[-L, L]$ of (1.1) with $u' \in C([-L, L], [-\infty, \infty])$ is called *classical* if $\left|u'(\pm L)\right| < \infty$, and it is called *non-classical* if $u'(-L) = \infty$ or $u'(L) = -\infty$; see [8,12]. In this paper we simply consider classical solutions of (1.1). Notice that it can be shown that (see [1,12]) any non-trivial positive solution $u \in C^2(-L, L) \cap C[-L, L]$ of (1.1) is *concave* and *symmetric* on $[-L, L]$. Moreover, $u'(-L) = -u'(L)$.

For any fixed $p, L > 0$, we define the bifurcation diagram $C_{p,L}$ of (1.1) by

$$C_{p,L} = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that the bifurcation diagram $C_{p,L}$ is ⊃-shaped (see e.g. Fig. 1(i) depicted below) if there exists $\lambda^* > 0$ such that $C_{p,L}$ consists of a continuous curve with exactly one turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ where the bifurcation diagram $C_{p,L}$ turns to the left.

This research is motivated by very recent papers of Brubaker and Pelesko [3,4] and Pan and Xing [15]. Brubaker and Pelesko [3] studied the existence and multiplicity of positive solutions of the prescribed mean curvature problem (1.3) with an inverse square type nonlinearity $\tilde{f}(u) = (1-u)^{-2}$. The problem is a derived variant of a *canonical* model used in the modeling of electrostatic Micro-Electro Mechanical Systems (MEMS) device obeying the electrostatic Coulomb law with the Coulomb force satisfying the *inverse square law* with respect to the distance of the two charged objects, which is a function of the deformation variable (cf. [17, p. 1324]). The modeling of electrostatic MEMS device consists of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at $+1$. In (1.3), u is the unknown profile of the deflecting MEMS membrane, λ is the drop voltage between the ground plate and the deflecting membrane, and the term $|\nabla u|^2$ is called a fringing field (cf. [3,18]). We refer the reader to [3,19–21] for the mathematical analysis of the electrostatic MEMS problem (1.3). Notice that the physically relevant dimensions are $n = 1$ and $n = 2$. In the case for $n = 1$, Ω_L is a rectangular strip with two opposite edges at $x = \pm L$ fixed ($2L$ is the length of the strip) and the remaining two edges free, the deflection $u = u(x, y)$ may be assumed a function of x only. In the case for $n = 2$, Ω_L is a planar bounded domain with smooth boundary, and L is the characteristic length (diameter) of the domain. In particular, Ω_L could be a circular disk of radius L.

With general $p > 0$, (1.1) is a generalized MEMS problem under the assumption that the Coulomb force satisfies the *inverse p-th power law*with respect to the distance of the two charged objects, where $p > 0$ characterizes the *force strength*. See [22,23].

Brubaker and Pelesko [4] and Pan and Xing [15] studied global bifurcation diagrams and exact multiplicity of positive (classical) solutions for problem of (1.1) with $p = 2$. They independently proved that there exists $L^* > 0$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram $C_{2,L}$ consists of a (continuous) ⊃-shaped curve when $L \geq L^*$, and as L transitions from greater than or equal to L^* to less than L^* the upper branch of the bifurcation diagram $C_{2,L}$ splits into two parts. See Fig. 1 and see [4, Theorem 1.1] and [15, Theorem 1.1] for details. Note that Brubaker and Pelesko [4, Theorem 1.1] showed that $L^* \approx 0.3499676$ and they also gave some computational results; see [4, Fig. 2].

In this paper, we extend and improve their results by generalizing the nonlinearity $(1-u)^{-2}$ to $(1-u)^{-p}$ with general $p \in [1, \infty)$; see Theorem 2.1 stated below. Our results (Theorems 2.1 and 2.2) also answer an open question raised by Brubaker and Pelesko [4, section 4] on the (possible) extension of (global) bifurcation diagram results of generalized MEMS problem (1.1). We find and prove that global bifurcation diagrams $C_{p,L}$ for $0 < p < 1$ are different to and more complicated than those for $p \geq 1$; compare Fig. 2 depicted below with Fig. 1. Thus p is also a bifurcation parameter to prescribed mean