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Resonance phenomenon for a Gelfand-type problem☆

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ABSTRACT

We consider positive radially symmetric solutions of

$$-\Delta u = \lambda(e^u - 1), \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B,$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. We establish infinite multiplicity of regular solutions for $3 \leq N \leq 9$ and some λ, and we obtain a bound for the Morse index and the number of solutions when $N \geq 10$.

1. Introduction

In this article, we are interested in the structure of the solution set of the boundary value problem

$$\begin{cases} -\Delta u = \lambda(e^u - 1), \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. Smooth solutions to (1.1) are radially symmetric and decreasing by the classical result of Gidas, Ni and Nirenberg [1].

Problem (1.1) is related to the following *Gelfand* problem:

$$\begin{cases} -\Delta u = \lambda e^u, & \text{in } B; \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.2}$$

Barenblatt [2] and Joseph and Lundgren [3], using phase-plane analysis, gave a complete description of the classical solutions to (1.2), which are again radially symmetric [1].

Proposition 1.1. *Assume $N \geq 1$, then there exists $\lambda^* = \lambda^*(N) > 0$, such that*

- *for $0 < \lambda < \lambda^*$, (1.2) has the minimal solution u_λ;*
- *for $\lambda = \lambda^*$, (1.2) has a unique solution;*
- *for $\lambda > \lambda^*$, (1.2) has no solution (even in the weak sense).*

Moreover, we have the following.

(a) *if $N = 1, 2$, then for $0 < \lambda < \lambda^*$, there are exactly two solutions to (1.2), one of them is the minimal solution u_λ. The other one, denoted by U_λ, has Morse index 1.*

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(b) *If* $3 \le N \le 9$, *then* $\lambda^* > 2(N-2)$. *For* $0 < \lambda < \lambda^*$, $\lambda \ne 2(N-2)$, (1.2) *has finitely many solutions; for* $\lambda = 2(N-2)$, (1.2) *has infinitely many solutions; for* λ *close to* $2(N-2)$, (1.2) *has a large number of solutions that converge to* $-2\log|x|$.

(c) *If* $N \ge 10$, *then* $\lambda^* = 2(N-2)$ *and* $u_* = -2\log|x|$. *Moreover* (1.2) *has a unique minimal solution* u_λ *for each* $\lambda \in (0, \lambda^*)$.

Nagasaki and Suzuki [4] classified the solutions of (1.2) according to their Morse index. In a few words, the family of regular solutions of (1.2) can be described as a curve $(u(s), \lambda(s))$ with $s \in [0, \infty)$, such that $(u(s), \lambda(s)) \to (0, 0)$ as $s \to 0$ and $(u(s), \lambda(s)) \to (u_\sigma, \lambda_\sigma)$ as $s \to \infty$, where $u_\sigma(r) = -2\log(r)$, $\lambda_\sigma = 2(N-2)$ is a singular solution of (1.2). In dimensions $3 \le N \le 9$, $\lambda(s)$ oscillates around $2(N-2)$ as $s \to \infty$ and the Morse index of $u(s)$ increases by one in each oscillation. In dimensions $N \ge 10$, $\lambda(s)$ is monotone, $u(s)$ is monotone and is stable for each s. We refer the reader to the book of L. Dupaigne [5] for further references on problem (1.2). Moreover, Berchio, Gazzola and Pierotti in [6] studied Gelfand type elliptic problems under Steklov boundary conditions.

A problem analogous to (1.1) is

$$\begin{cases} -\Delta u = u^p + \lambda u, \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.3}$$

where $p > 1$ and $\lambda > 0$ is a parameter. According to classical bifurcation theory [7], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathcal{C}$ of solutions of (1.3), where μ_1 is the first eigenvalue of the negative Laplacian operator under Dirichlet boundary condition in B.

- If $p < \frac{N+2}{N-2} (N \ge 3)$, for $\lambda < \mu_1$, there is a positive solution of (1.3) by a standard constrained minimization procedure involving compactness of the Sobolev embedding. Moreover, by Pohozaev's identity [8], problem (1.3) has no solutions for $\lambda \le 0$ whenever $p \ge \frac{N+2}{N-2}$.
- If $p = \frac{N+2}{N-2}$, which is the classical Brezis–Nirenberg problem [9], problem (1.3) has a solution for $0 < \lambda < \mu_1$ if $N \ge 4$, and for $\frac{1}{4}\mu_1 < \lambda < \mu_1$ if $N = 3$.
- If $p > \frac{N+2}{N-2}$, Dolbeault and Flores found that if $p > \frac{N+2}{N-2}$, and $p < \frac{N-2\sqrt{N-1}}{N-2\sqrt{N-1}-4}$ or $N \le 10$, then there is a unique number $\lambda_* > 0$, such that for λ close to λ_*, a large number of classical solutions of (1.3) exist. In particular, there are infinitely many classical solutions for $\lambda = \lambda_*$. Recently, Guo and Wei in [10] showed that the structure of the branch $\mathcal{C}$ changes for $p \ge p_c$ and $\frac{N+2}{N-2} < p < p_c$, where $p_c = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}$ if $N \ge 11$; and $p_c = \infty$ if $2 \le N \le 10$. Moreover, they established that for $\frac{N+2}{N-2} < p < p_c$, $\mathcal{C}$ turns infinitely many times around $\lambda_* \in (0, \mu_1)$. For $p \ge p_c$, all solutions have a finite Morse index, and for $N \ge 12$ and $p > p_c$ sufficiently large all solutions have exactly Morse index one.

This paper is devoted to the study of the structure of solutions to problem (1.1). We start with some general remarks. First, classical solutions of (1.1) can exist only for λ in some interval.

Proposition 1.2. *Let* μ_1 *be the first eigenvalue of the* $-\Delta$ *under Dirichlet boundary condition in B. Then there exists* $\lambda_0 > 0$, *such that a necessary condition for existence of classical solutions to problem* (1.1) *is* $\lambda \in (\lambda_0, \mu_1)$.

See a proof in the Appendix. By classical bifurcation theory [11,7] we have that $(\mu_1, 0)$ is a bifurcation point of solutions to (1.1). Both observations are also valid if we replace the ball by a bounded smooth domain (star shaped in the case of Proposition 1.2).

We are interested also in weak solutions, allowing for possible singularities.

Definition 1.3. We say that $u \in H_0^1(B)$ is a weak solution of (1.1) if $e^u \in L^1(B)$ and

$$\int_B \nabla u \nabla \varphi = \lambda \int_B (e^u - 1)\varphi \quad \text{for all } \varphi \in C_0^\infty(B). \tag{1.4}$$

We say that a weak solution u of (1.1) is regular (resp., singular) if $u \in L^\infty(B)$ (resp., $u \notin L^\infty(B)$).

We say that a radial weak solution u of (1.1) is a weakly singular solution if it is singular and $\lim_{r\to 0} ru'(r)$ exists.

We first study singular solutions to (1.1).

Theorem 1.4. *Assume* $N \ge 3$. *Let* $\lambda > 0$ *and suppose that* $u \in C^2(B\backslash\{0\})$, $u \ge 0$ *is a radial solution of*

$$-\Delta u = \lambda(e^u - 1) \quad \text{in } B\backslash\{0\}. \tag{1.5}$$

Then either

(a) *u can be extended as a function in* $C^\infty(B)$ *and* (1.5) *holds in B,*